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# ROBUST THREE-AXIS ATTITUDE STABILIZATION FOR INERTIAL POINTING SPACECRAFT USING MAGNETORQUERS

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In this work feedback control laws are designed for achieving three-axis attitude stabilization of inertial pointing spacecraft using only magnetic torquers. The designs are based on an almost periodic model of geomagnetic field along the spacecraft's orbit. Both attitude plus attitude rate feedback, and attitude only feedback are proposed. Both feedback laws achieve local exponential stability robustly with respect to large uncertainties in the spacecrafts inertia matrix. The latter properties are proved using general averaging and Lyapunov stability. Simulations are included to validate the effectiveness of the proposed control algorithms.

## INTRODUCTION

The focus of this paper is three-axis attitude stabilization for inertial pointing spacecraft using only magnetic torquers as actuators. Magnetic coils are widely used on low Earth-orbit small satellites to implement attitude control actuators due to their lightweight and low power requirements; in fact, these actuators do not need propellent except for electricity that can be generated on board of the spacecraft. However, magnetotorquers have the important limitation that control torque is constrained to belong to the plane orthogonal to the geomagnetic field; this is a consequence of the fact that magnetic torque is equal to the cross product of the magnetic moment vector (generated by currents flowing through magnetic coils) and geomagnetic field vector. In spite of the described situation three-axis attitude stabilization using only magnetic torque is still possible because of the time-variability of geomagnetic field along the spacecraft's orbit.

A considerable amount of work has been dedicated to the design of magnetic control laws; a survey of several methods presented in the literature is reported in Reference 1. Most of works are based on periodic approximation of the time-variation of the geomagnetic field along the orbit (see e.g. References 2, 3, 4, 5), and in such setting stability and disturbance attenuation have been proved using results from linear periodic systems. In References 6 and 7 stability has been achieved even when a non periodic, and thus more accurate, approximation of the geomagnetic field is adopted; in both works feedback control laws that require measures of both attitude and attitude-rate (i.e. state feedback control laws ) are proposed; moreover, in Reference 6 feedback control algorithms which need measures of attitude only (i.e. output feedback control algorithms) are presented, too. All the control algorithms in References 6 and 7 require exact knowledge of the spacecraft's inertia matrix; however, the inertia matrix of a spacecraft is often subject to large uncertainties since its accurate

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determination is not easy; as a result, it is important to determine control algorithms which achieve attitude stabilization in spite of those uncertainties.

In this work we present control laws obtained by modifying those in References 6 and 7, which achieve local exponential stability in spite of large uncertainties on the inertia matrix. The latter results are derived adopting an almost periodic model of the geomagnetic field along the spacecraft's orbit. As in References 6 and 7 the main tools used in the proofs are general averaging and Lyapunov stability (see Reference 8).

### Notations

For $x \in \mathbb{R}^n$, $\|x\|$ denotes the Eucledian norm of $x$; for a square matrix $A$, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalue of $A$ respectively; $\|A\|$ denotes the 2-norm of $A$ which is equal to $\|A\| = [\lambda_{max}(A^T A)]^{1/2}$. Symbol $I$ represents the identity matrix. For $a \in \mathbb{R}^3$, $a^\times$ represents the skew symmetric matrix

$$a^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{1}$$

so that for $b \in \mathbb{R}^3$, the multiplication $a^\times b$ is equal to the cross product $a \times b$.

## MODELING

In order to describe the attitude dynamics of an Earth-orbiting rigid spacecraft, and in order to represent the geomagnetic field, it is useful to introduce the following reference frames.

1. *Earth-centered inertial frame* $\mathcal{F}_i$ . A commonly used inertial frame for Earth orbits is the Geocentric Equatorial Frame, whose origin is in the Earth's center, its $x_i$ axis is the vernal equinox direction, its $z_i$ axis coincides with the Earth's axis of rotation and points northward, and its $y_i$ axis completes an orthogonal right-handed frame (see Reference 9 Section 2.6.1).
2. *Spacecraft body frame* $\mathcal{F}_b$. The origin of this right-handed orthogonal frame attached to the spacecraft, coincides with the satellite's center of mass; its axis are chosen so that the inertial pointing objective is having $\mathcal{F}_b$ aligned with $\mathcal{F}_i$.

Since the inertial pointing objective consists in aligning $\mathcal{F}_b$ to $\mathcal{F}_i$, the focus will be on the relative kinematics and dynamics of the satellite with respect to the inertial frame. Let

$$q = [q_1\ q_2\ q_3\ q_4]^T = [q_v^T\ q_4]^T$$

with $\|q\| = 1$ be the unit quaternion representing rotation of $\mathcal{F}_b$ w.r.t. $\mathcal{F}_i$ ; then, the corresponding attitude matrix is given by

$$A(q) = (q_4^2 - q_v^T q_v)I + 2q_v q_v^T - 2q_4 q_v^\times \tag{2}$$

(see Reference 10 Section 5.4).

Let

$$W(q) = \frac{1}{2} \begin{bmatrix} q_4 I + q_v^\times \\ -q_v^T \end{bmatrix} \tag{3}$$

Then the relative attitude kinematics is given by

$$\dot{q} = W(q)\omega \tag{4}$$

where $\omega \in \mathbb{R}^3$ is the angular rate of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ resolved in $\mathcal{F}_b$ (see Reference 10 Section 5.5.3).

The attitude dynamics in body frame can be expressed by

$$J\dot{\omega} = -\omega^{\times} J\omega + T \tag{5}$$

where $J \in \mathbb{R}^{3\times 3}$ is the spacecraft inertia matrix and $T \in \mathbb{R}^3$ is the vector of external torque expressed in $\mathcal{F}_b$ (see Reference 10 Section 6.4). As stated in the introduction, here we consider $J$ uncertain since in practice it is often difficult to determine accurately the inertia matrix of a satellite; however, we require to know a lower bound for the smallest principal moment of inertia and an upper bound for the largest one. Thus, the following assumption on $J$ is made.

**Assumption 1.** *The inertia matrix $J$ is unknown but there exist known bounds $0 < J_{min} \leq J_{max}$ such that the following hold*

$$0 < J_{min} \leq \lambda_{min}(J) \leq \lambda_{max}(J) = \|J\| \leq J_{max} \tag{6}$$

The spacecraft is equipped with three magnetic coils aligned with the $\mathcal{F}_b$ axes which generate the magnetic attitude control torque

$$T = m_{coils} \times B^b = -B^{b\times}\, m_{coils} \tag{7}$$

where $m_{coils} \in \mathbb{R}^3$ is the vector of magnetic moments for the three coils, and $B^b$ is the geomagnetic field at spacecraft expressed in body frame $\mathcal{F}_b$. From the previous equation, we see that magnetic torque can only be perpendicular to geomagnetic field.

Let $B^i$ be the geomagnetic field at spacecraft expressed in inertial frame $\mathcal{F}_i$. Note that $B^i$ varies with time both because of the spacecraft's motion along the orbit and because of time variability of the geomagnetic field. Then

$$B^b(q,t) = A(q)B^i(t)$$

which shows explicitly the dependence of $B^b$ on both $q$ and $t$.

Grouping together equations (4) (5) (7) the following nonlinear time-varying system is obtained

$$\begin{array}{rcl} \dot{q} & = & W(q)\omega \\ J\dot{\omega} & = & -\omega^{\times} J\omega - B^b(q,t)^{\times}\, m_{coils} \end{array} \tag{8}$$

in which $m_{coils}$ is the control input.

For control design, it is important to characterize the time-dependence of $B^b(q,t)$ which is the same as characterizing the time-dependence of $B^i(t)$. In this work we adopt the following dipole model of the geomagnetic field expressed in $\mathcal{F}_i$ (see Reference 11 Appendix H)

$$B^i(r^i,t) = \frac{\mu_m}{\|r^i\|^3}[3(\hat{m}^i(t)^T \hat{r}^i)\hat{r}^i - \hat{m}^i(t)] \tag{9}$$

In equation (9), $\mu_m$ is the total dipole strength, $r^i$ is the position vector resolved in $\mathcal{F}_i$, of the point where the field is computed, and $\hat{r}^i$ is the vector of the direction cosines of $r^i$; finally $\hat{m}^i(t)$ is the vector of the direction cosines of the Earth's magnetic dipole expressed in $\mathcal{F}_i$ which is set equal to

$$\hat{m}^i(t) = \begin{bmatrix} \sin(\theta_m)\cos(\omega_e t + \alpha_0) \\ \sin(\theta_m)\sin(\omega_e t + \alpha_0) \\ \cos(\theta_m) \end{bmatrix} \tag{10}$$

where $\theta_m$ is the coelevation of the dipole, $\omega_e = 360.99$ deg/day is the average rotation rate of the Earth, and $\alpha_0$ is the right ascension of the dipole at time $t = 0$; clearly, in equation (10) Earth's rotation has been taken into account. It has been obtained that for year 2010 $\mu_m = 7.746\ 10^{15}$ Wb m and $\theta_m = 170.0°$ (see Reference 12); then, the Earth's magnetic dipole is tilted with respect to Earth's axis of rotation.

Here, we are interested in determining $B^i$ along the spacecraft's orbit. Assume that the orbit is circular, and define a coordinate system $x_p$, $y_p$ in the orbital's plane whose origin is at Earth's center; then, the position of satellite's center of mass is clearly given by

$$\begin{aligned} x^p(t) &= R\cos(nt + \phi_0) \\ y^p(t) &= R\sin(nt + \phi_0) \end{aligned} \tag{11}$$

where $R$ is the radius of the circular orbit, $n$ is the orbital rate, and $\phi_0$ an initial phase . Let $incl$ be the orbit's inclination and $\Omega$ be the right ascension of the ascending node; then, coordinates of the satellite's center of mass in inertial frame $\mathcal{F}_i$ can be obtained as follows (see Reference 9 Section 2.6.2)

$$r^i(t) = R_z(-\Omega)R_x(-incl)\begin{bmatrix} x^p(t) \\ y^p(t) \\ 0 \end{bmatrix} \tag{12}$$

where

$$R_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) \\ 0 & -\sin(\varphi) & \cos(\varphi) \end{bmatrix} \tag{13}$$

corresponds to a rotation around the $x$-axis of magnitude $\varphi$ and

$$R_z(\psi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

corresponds to a rotation around the $z$-axis of magnitude $\psi$ .

Combining equations (9) (10) (11) (12), the expression of $B^i(t)$ can be obtained, and it is easy to see that $B^i(t)$ is a linear combination of sinusoidal functions of $t$ having different frequencies. As a result, $B^i(t)$ is an almost periodic function of $t$ (see Reference 8 Section 10.6), and consequently system (8) is an almost periodic nonlinear system.

## CONTROL DESIGN

As stated before, the control objective is driving the spacecraft so that $\mathcal{F}_b$ is aligned with $\mathcal{F}_i$. From (2) it follows that $A(q) = I$ for $q = [q_v^T\ q_4]^T = \pm\bar{q}$ where $\bar{q} = [0\ 0\ 0\ 1]^T$. Thus, the objective is designing control strategies for $m_{coils}$ so that $q_v \to 0$ and $\omega \to 0$. Here we will present feedback laws that locally exponentially stabilize equilibrium $(q, \omega) = (\bar{q}, 0)$.

First, since $B^b$ can be measured using magnetometers, apply the following preliminary control which guarantees that $m_{coils}$ is orthogonal to $B^b$

$$m_{coils} = B^b(q,t) \times u = B^b(q,t)^{\times} u = -(B^b(q,t)^{\times})^T u \tag{15}$$

where $u \in \mathbb{R}^3$ is a new control vector. Then, it holds that

$$\begin{array}{rcl} \dot{q} & = & W(q)\omega \\ J\dot{\omega} & = & -\omega^{\times} J\omega + \Gamma^b(q,t)u \end{array} \tag{16}$$

where

$$\Gamma^b(q,t) = (B^b(q,t)^{\times})(B^b(q,t)^{\times})^T = B^b(q,t)^T B^b(q,t) I - B^b(q,t) B^b(q,t)^T \tag{17}$$

Let

$$\Gamma^i(t) = (B^i(t)^{\times})(B^i(t)^{\times})^T = B^i(t)^T B^i(t) I - B^i(t) B^i(t)^T \tag{18}$$

then it is easy to verify that

$$\Gamma^b(q,t) = A(q)\Gamma^i(t)A(q)^T$$

so that (16) can be written as

$$\begin{array}{rcl} \dot{q} & = & W(q)\omega \\ J\dot{\omega} & = & -\omega^{\times} J\omega + A(q)\Gamma^i(t)A(q)^T u \end{array} \tag{19}$$

Since $B^i(t)$ is a linear combination of sinusoidal functions of $t$ having different frequencies, so is $\Gamma^i(t)$. As a result, the following average

$$\Gamma^i_{av} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Gamma^i(\tau) d\tau \tag{20}$$

is well defined (see Reference 8 Section 10.6). Consider the following assumption on $\Gamma^i_{av}$ .

**Assumption 2.** *The spacecraft's orbit satisfies condition* $\Gamma^i_{av} > 0$.

**Remark 1.** Since $\Gamma^i(t) \geq 0$ (see (18)), Assumption 2 is equivalent to requiring that $\det(\Gamma^i_{av}) \neq 0$. The expression of $\det(\Gamma^i_{av})$ based on the model of the geomagnetic field presented before is quite complex, and it is not easy to get an insight from it; however, if coelevation of Earth's magnetic dipole $\theta_m = 170.0°$ is approximated to $\theta_m = 180°$ deg, which corresponds to having Earth's magnetic dipole aligned with its rotation axis, then the geomagnetic field in a fixed point of the inertial space becomes constant with respect to time (see (9) and (10)); consequently $B^i(t)$, which represents the geomagnetic field along the orbit, becomes periodic, and the expression of $\det(\Gamma^i_{av})$ simplifies as follows

$$\det(\Gamma^i_{av}) = \frac{9\mu_m^6}{1024\, R^{18}}[345 - 92\cos(2\, incl) + 3\cos(4\, incl)] \sin(incl)^2 \tag{21}$$

Thus, in such simplified scenario issues on fulfillment of Assumption 2 arise only for low inclination orbits.

### State feedback

In this subsection, a stabilizing static state (attitude and attitude rate) feedback for system (19) is presented. It is obtained as a simple modification of the one proposed in Reference 7 (Proposition 10). The important property that is achieved through such modification is robustness with respect to uncertainties on the inertia matrix; that is, the modified control algorithm achieves stabilization for all $J$'s that fulfill Assumption 1.

**Theorem 2.** *Consider the magnetically actuated spacecraft described by (19) with uncertain inertia matrix $J$ satisfying Assumption 1. Apply the following proportional derivative control law*

$$u = -(\epsilon^2 k_1 q_v + \epsilon k_2 \omega) \tag{22}$$

*with $k_1 > 0$ and $k_2 > 0$. Then, under Assumption 2, there exists $\epsilon^* > 0$ such that for any $0 < \epsilon < \epsilon^*$, equilibrium $(q, \omega) = (\bar{q}, 0)$ is locally exponentially stable for (19) (22).*

*Proof.* In order to prove local exponential stability of equilibrium $(q, \omega) = (\bar{q}, 0)$, it suffices considering the restriction of (19) (22) to the open set $\mathbb{S}^{3+} \times \mathbb{R}^3$ where

$$\mathbb{S}^{3+} = \{q \in \mathbb{R}^4 \mid \|q\| = 1,\ q_4 > 0\} \tag{23}$$

On the latter set the following holds

$$q_4 = (1 - q_v^T q_v)^{1/2} \tag{24}$$

Consequently, the restriction of (19) (22) to $\mathbb{S}^{3+} \times \mathbb{R}^3$ is given by the following reduced order system

$$\begin{array}{rcl} \dot{q}_v & = & W_v(q_v)\omega \\ J\dot{\omega} & = & -\omega^\times J\omega - A_v(q_v)\Gamma^i(t)A_v(q_v)^T(\epsilon^2 k_1 q_v + \epsilon k_2 \omega) \end{array} \tag{25}$$

where

$$W_v(q_v) = \frac{1}{2}\left[\left(1 - q_v^T q_v\right)^{1/2} I + q_v^\times\right] \tag{26}$$

and

$$A_v(q_v) = \left(1 - 2q_v^T q_v\right) I + 2q_v q_v^T - 2\left(1 - q_v^T q_v\right)^{1/2} q_v^\times \tag{27}$$

Consider the linear approximation of (25) around $(q_v, \omega) = (0, 0)$ which is given by

$$\begin{array}{rcl} \dot{q}_v & = & \frac{1}{2}\omega \\ \dot{\omega} & = & -J^{-1}\Gamma^i(t)(\epsilon^2 k_1 q_v + \epsilon k_2 \omega) \end{array} \tag{28}$$

Introduce the following state-variables' transformation

$$z_1 = q_v \quad z_2 = \omega/\epsilon$$

with $\epsilon > 0$ so that system (25) is transformed into

$$\begin{array}{rcl} \dot{z}_1 & = & \frac{\epsilon}{2} z_2 \\ \dot{z}_2 & = & -\epsilon J^{-1}\Gamma^i(t)(k_1 z_1 + k_2 z_2) \end{array} \tag{29}$$

Rewrite system (29) in the following matrix form

$$\dot{z} = \epsilon A(t) z \tag{30}$$

where

$$A(t) = \begin{bmatrix} 0 & \frac{1}{2}I \\ -k_1 J^{-1}\Gamma^i(t) & -k_2 J^{-1}\Gamma^i(t) \end{bmatrix}$$

and consider the so called "average system" of (30)

$$\dot{z} = \epsilon A_{av} z \tag{31}$$

with

$$A_{av} = \begin{bmatrix} 0 & \frac{1}{2}I \\ -k_1 J^{-1}\Gamma^i_{av} & -k_2 J^{-1}\Gamma^i_{av} \end{bmatrix}$$

and where $\Gamma^i_{av}$ was defined in (20).

We will show that after having performed an appropriate coordinate transformation system (30) can be seen as a perturbation of (31) (see Reference 8 Section 10.4). For that purpose note that since $\Gamma^i(t)$ is a linear combination of sinusoidal functions of $t$ having different frequencies, then there exists $k_\Delta > 0$ such that the following holds (see Reference 8 Section 10.6)

$$\left\| \frac{1}{T}\int_0^T \Gamma^i(\tau)d\tau - \Gamma^i_{av} \right\| \leq k_\Delta \frac{1}{T+1} \quad \forall T > 0$$

Let

$$\Delta(t) = \int_0^t (\Gamma^i(\tau) - \Gamma^i_{av}) d\tau$$

then for $t > 0$

$$\|\Delta(t)\| = t \left\| \left[ \frac{1}{t}\int_0^t \Gamma^i(\tau)d\tau - \Gamma^i_{av} \right] \right\| \leq k_\Delta \frac{t}{t+1}$$

hence

$$\|\Delta(t)\| \leq k_\Delta \quad \forall t \geq 0 \tag{32}$$

Let

$$U(t) = \int_0^t [A(\tau) - A_{av}]d\tau = \begin{bmatrix} 0 & 0 \\ -k_1 J^{-1}\Delta(t) & -k_2 J^{-1}\Delta(t) \end{bmatrix} \tag{33}$$

and observe that there there exists $a > 0$ such that the following holds

$$\|U(t)\| \leq a(k_1 + k_2)\|J^{-1}\| \; \|\Delta(t)\| \quad \forall t \geq 0 \tag{34}$$

Observe that from (6) it follows that

$$\|J^{-1}\| = \frac{1}{\lambda_{min}(J)} \leq \frac{1}{J_{min}}$$

thus

$$\|U(t)\| \leq \frac{a(k_1 + k_2)k_\Delta}{J_{min}} \quad \forall t \geq 0 \tag{35}$$

Now consider the transformation matrix

$$T(t,\epsilon) = I + \epsilon U(t) = \begin{bmatrix} I & 0 \\ -\epsilon k_1 J^{-1}\Delta(t) & I - \epsilon k_2 J^{-1}\Delta(t) \end{bmatrix} \tag{36}$$

Since (35) holds, then if $\epsilon$ is small enough, then $T(t,\epsilon)$ is non singular for all $t \geq 0$. Thus, we can define the coordinate transformation

$$w = T(t,\epsilon)^{-1} z$$

In order to compute the state equation of system (30) in the new coordinates it is convenient to consider the inverse transformation

$$z = T(t,\epsilon) w$$

and differentiate with respect to time both sides obtaining

$$\epsilon A(t) T(t,\epsilon) w = \frac{\partial T}{\partial t}(t,\epsilon) w + T(t,\epsilon)\dot{w}$$

Consequently

$$\dot{w} = T(t,\epsilon)^{-1} \left[ \epsilon A(t) T(t,\epsilon) - \frac{\partial T}{\partial t}(t,\epsilon) \right] w \tag{37}$$

Observe that

$$T(t,\epsilon)^{-1} = \begin{bmatrix} I & 0 \\ +[I - \epsilon k_2 J^{-1}\Delta(t)]^{-1} \epsilon k_1 J^{-1}\Delta(t) & [I - \epsilon k_2 J^{-1}\Delta(t)]^{-1} \end{bmatrix}$$

Thus, note that using the mean value theorem with respect to $\epsilon$, matrix $[I - \epsilon k_2 J^{-1}\Delta(t)]^{-1}$ can be expressed as follows

$$[I - \epsilon k_2 J^{-1}\Delta(t)]^{-1} = I + \epsilon k_2 [I - \tilde{\epsilon} k_2 J^{-1}\Delta(t)]^{-2} J^{-1}\Delta(t)$$

where $\tilde{\epsilon} \in (0\ \epsilon)$. As a result $T(t,\epsilon)^{-1}$ can be written as

$$T(t,\epsilon)^{-1} = I + \epsilon S(t,\epsilon) \tag{38}$$

with

$$S(t,\epsilon) = \begin{bmatrix} 0 & 0 \\ +[I - \epsilon k_2 J^{-1}\Delta(t)]^{-1} k_1 J^{-1}\Delta(t) & [I - \tilde{\epsilon} k_2 J^{-1}\Delta(t)]^{-2} k_2 J^{-1}\Delta(t) \end{bmatrix}$$

Observe that for $\epsilon$ sufficiently small $S(t,\epsilon)$ is bounded for all $t \geq 0$. Moreover, from (33) and (36) obtain the following

$$\frac{\partial T}{\partial t}(t,\epsilon) = \epsilon \frac{\partial U}{\partial t}(t,\epsilon) = \epsilon (A(t) - A_{av}) \tag{39}$$

Then, from (36) (37) (38) (39) we obtain

$$\dot{w} = \epsilon [A_{av} + \epsilon H(t,\epsilon)] w \tag{40}$$

where

$$H(t,\epsilon) = A(t)U(t) + S(t,\epsilon)A_{av} + \epsilon S(t,\epsilon) A(t) U(t)$$

Thus we have shown that in coordinates $w$ system (30) is a perturbation of system (31); moreover, clearly, for the perturbation factor $H(t,\epsilon)$ it occurs that for $\epsilon$ small enough there exists $k_H > 0$ such that

$$\|H(t,\epsilon)\| \leq k_H \quad \forall t \geq 0 \tag{41}$$

Let us focus on system

$$\dot{w} = A_{av} w \tag{42}$$

which in expanded form reads as follows

$$\begin{aligned} \dot{w}_1 &= \frac{1}{2} w_2 \\ J\dot{w}_2 &= -\Gamma^i_{av}(k_1 w_1 + k_2 w_2) \end{aligned} \tag{43}$$

Consider the candidate Lyapunov function for system (43) (see References 13 and 14)

$$V(w_1, w_2) = k_1 w_1^T \Gamma^i_{av} w_1 + 2\beta w_1^T J w_2 + \frac{1}{2} w_2^T J w_2$$

with $\beta > 0$. Note that

$$\begin{aligned} V(w_1,w_2) &\geq k_1 \lambda_{min}(\Gamma^i_{av})\|w_1\|^2 - 2\beta\|J\|\|w_1\|\|w_2\| + \frac{1}{2}\lambda_{min}(J)\|w_2\|^2 \\ &\geq \left(k_1\lambda_{min}(\Gamma^i_{av}) - \beta J_{max}\right)\|w_1\|^2 + \left(\frac{1}{2}J_{min} - \beta J_{max}\right)\|w_2\|^2 \end{aligned}$$

Thus for $\beta$ small enough, $V$ is positive definite. Moreover, it is easy to show that

$$V(w_1,w_2) \leq (k_1\|\Gamma^i_{av}\| + \beta J_{max})\|w_1\|^2 + \left(\frac{1}{2} + \beta\right) J_{max}\|w_2\|^2$$

Regarding $\dot{V}$ the following holds

$$\begin{aligned} &\dot{V}(w_1,w_2) = -2\beta k_1 w_1^T \Gamma^i_{av} w_1 - 2\beta k_2 w_1^T \Gamma^i_{av} w_2 - k_2 w_2^T \Gamma^i_{av} w_2 + \beta k_2 w_2^T J w_2 \\ &\leq -2\beta k_1 \lambda_{min}(\Gamma^i_{av})\|w_1\|^2 - 2\beta k_2\|\Gamma^i_{av}\|\|w_1\|\|w_2\| - k_2\lambda_{min}(\Gamma^i_{av})\|w_2\|^2 + \beta k_2\|J\|\|w_2\|^2 \end{aligned}$$

Use the following Young's inequality

$$-2\|w_1\|\|w_2\| \leq \frac{k_1\lambda_{min}(\Gamma^i_{av})}{k_2\|\Gamma^i_{av}\|}\|w_1\|^2 + \frac{k_2\|\Gamma^i_{av}\|}{k_1\lambda_{min}(\Gamma^i_{av})}\|w_2\|^2$$

so to obtain

$$\dot{V}(w_1,w_2) \leq -\beta\lambda_{min}(\Gamma^i_{av})\|w_1\|^2 - \left[k_2\lambda_{min}(\Gamma^i_{av}) + \beta\left(\frac{k_2^2\|\Gamma^i_{av}\|}{k_1\lambda_{min}(\Gamma^i_{av})} - J_{max}\right)\right]\|w_2\|^2 \tag{44}$$

Thus, for $\beta$ small enough $\dot{V}$ is negative definite and system (42) is exponentially stable for all $J$'s satisfying Assumption 1. For the sequel it is useful to rewrite the Lyapunov function $V$ in the following compact form

$$V(w_1,w_2) = w^T P w$$

where clearly

$$P = \begin{bmatrix} k_1\Gamma^i_{av} & \beta J \\ \beta J & \frac{1}{2}J \end{bmatrix}$$

Then, note that the following holds

$$\|P\| \leq k_P \tag{45}$$

with

$$k_P = a\left[\frac{1}{2}k_1\|\Gamma^i_{av}\| + \left(2\beta + \frac{1}{2}\right)J_{max}\right] \tag{46}$$

where constant $a > 0$ was introduced in equation (34). Moreover from equation (44) it follows immediately that there exists $k_V > 0$ such that

$$2w^T P A_{av} w \leq -k_V\|w\|^2 \tag{47}$$

Now for system (40) consider the same Lyapunov function $V$ used for system (42); the derivative of $V$ along the trajectories of (40) is given by

$$\dot{V}(w_1, w_2) = \epsilon[2w^T P A_{av} w + 2\epsilon w^T P H(t,\epsilon) w]$$

Thus, using (41) (45) (47) we obtain that for $\epsilon$ small enough the following holds

$$\dot{V}(w_1, w_2) \leq \epsilon[-k_V + 2\epsilon k_P k_H]\|w\|^2$$

Thus for $\epsilon$ sufficiently small system (40) is exponentially stable. As a result, for the same values of $\epsilon$ equilibrium $(q_v, \omega) = (0, 0)$ is locally exponentially stable for (28), and consequently for the nonlinear system (25), too. From equation (24) it follows that given $d < 1$, there exists $L > 0$ such that

$$|q_4 - 1| \leq L\|q_v\| \quad \forall\ \|q_v\| < d$$

Thus, exponential stability of $(q, \omega) = (\bar{q}, 0)$ for (19) (22) can be easily obtained.

$\square$

**Remark 3.** Given an inertia matrix $J$ it is relatively simple to show that there exists $\epsilon^* > 0$ such that setting $0 < \epsilon < \epsilon^*$ the closed-loop system (19) (22) is locally exponentially stable at $(q, \omega) = (\bar{q}, 0)$. It turns out that the value of $\epsilon^* > 0$ depends on $J$; consequently, if $J$ is uncertain, $\epsilon^*$ cannot be determined. However, the previous Theorem has shown that even in the case of unkown $J$, if bounds $J_{min}$ and $J_{max}$ on its principal inertia moments are known, then it is possible to determine an $\epsilon^* > 0$ such that picking $0 < \epsilon < \epsilon^*$ local exponential stability is guaranteed for all $J$'s that satisfy those bounds.

**Remark 4.** Assumption 2 represents an average controllability condition in the following sense. Note that, as a consequence of the fact that magnetic torques can only be perpendicular to the geomagnetic field, it occurs that matrix $\Gamma^i(t)$ is singular for each $t$ since $\Gamma^i(t)B^i(t) = 0$ (see (18)); thus, system (19) is not fully controllable at each time instant; as a result, having $\det(\Gamma^i_{av}) \neq 0$ can be interpreted as the ability in the average system to apply magnetic torques in any direction.

**Remark 5.** It is straightforward to show that the same stabilization result can be obtained for equilibrium $(q, \omega) = (-\bar{q}, 0)$ by using the same control law with $k_1 < 0$.

**Remark 6.** The obtained robust stability result hold even if saturation on the magnetic dipole moments is taken into account by replacing control (15) with

$$m_{coils} = m_{coils\ max}\ \text{sat}\left(\frac{1}{m_{coils\ max}} B^b(q,t) \times u\right) \tag{48}$$

where $m_{coils\ max}$ is the saturation limit on the magnetic dipole moments, and sat : $\mathbb{R}^3 \to \mathbb{R}^3$ is the standard saturation function defined as follows; given $x \in \mathbb{R}^3$, the $i$-th component of sat$(x)$ is equal to $x_i$ if $|x_1| \leq 1$, otherwise it is equal to either 1 or -1 depending if $x_i$ is positive or negative. The previous theorem still holds because saturation does not modify the linearized system (28).

**Remark 7.** In practical applications the values for gains $k_1$, $k_2$ can be chosen by trial and error following standard guidelines used in proportional derivative control. For selecting $\epsilon$ in principle we could proceed as follows; determine $\epsilon^*$ by following the procedure presented in the previous proof and pick $0 < \epsilon < \epsilon^*$. However, if it is too complicated to follow that method, an appropriate value for $\epsilon$ could be found by trial and error as well.

**Output feedback**

Being able to achieve stability without using attitude rate measures is important from a practical point of view since rate gyros consume power, and they increase cost and weight more than the devices needed to implement extra control logic.

In the following thorem we propose a robust dynamic output feedback control algorithm that is obtained as a simple modification of the output feedback presented in Reference 6.

**Theorem 8.** *Consider the magnetically actuated spacecraft described by (19) with uncertain inertia matrix $J$ satisfying Assumption 1. Apply the following dynamic attitude feedback control law*

$$\begin{aligned} \dot{\delta} &= \alpha(q - \epsilon\lambda\delta) \\ u &= -\epsilon^2\left(k_1 q_v + k_2\alpha\lambda W(q)^T(q - \epsilon\lambda\delta)\right) \end{aligned} \tag{49}$$

*with $\delta \in \mathbb{R}^4$, $k_1 > 0$, $k_2 > 0$, $\alpha > 0$, and $\lambda > 0$. Then, under Assumption 2, there exists $\epsilon^* > 0$ such that for any $0 < \epsilon < \epsilon^*$, equilibrium $(q, \omega, \delta) = (\bar{q}, 0, \frac{1}{\epsilon\lambda}\bar{q})$ is locally exponentially stable for (19) (49).*

*Proof.* In order to prove local exponential stability of equilibrium $(q, \omega, \delta) = (\bar{q}, 0, \frac{1}{\epsilon\lambda}\bar{q})$, it suffices considering the restriction of (19) (49) to the open set $\mathbb{S}^{3+} \times \mathbb{R}^3 \times \mathbb{R}^4$ where $\mathbb{S}^{3+}$ was defined in (23); the latter restriction is given by the following reduced order system

$$\begin{aligned} \dot{q}_v &= W_v(q_v)\omega \\ J\dot{\omega} &= \omega^\times J\omega - \epsilon^2 A_v(q_v)\Gamma^i(t)A_v(q_v)^T\left(k_1 q_v + k_2\alpha\lambda W_r(q_v)^T\left(\begin{bmatrix} q_v \\ (1 - q_v^T q_v)^{1/2} \end{bmatrix} - \epsilon\lambda\delta\right)\right) \\ \dot{\delta} &= \alpha(q - \epsilon\lambda\delta) \end{aligned} \tag{50}$$

where $W_v$ and $A_v$ were defined in equations (26) and (27) respectively and $W_r(q_v)$ is equal to

$$W_r(q_v) = \frac{1}{2}\begin{bmatrix} (1 - q_v^T q_v)^{1/2} I + q_v^\times \\ -q_v^T \end{bmatrix}$$

Partition $\delta \in \mathbb{R}^4$ as follows

$$\delta = [\delta_v^T \ \delta_4]^T$$

where clearly $\delta_v \in \mathbb{R}^3$, and consider the linear approximation of (50) around $(q_v, \omega, \delta_v, \delta_4) = (0, 0, 0, \frac{1}{\epsilon\lambda})$ which is given by

$$\begin{aligned}
\dot{q}_v &= \frac{1}{2}\omega \\
J\dot{\omega} &= -\epsilon^2\Gamma^i(t)\left(k_1 q_v + \frac{1}{2}k_2\alpha\lambda(q_v - \epsilon\lambda\delta_v)\right) \\
\dot{\delta}_v &= \alpha(q_v - \epsilon\lambda\delta_v) \\
\dot{\tilde{\delta}}_4 &= -\alpha\epsilon\lambda\tilde{\delta}_4
\end{aligned} \tag{51}$$

where $\tilde{\delta}_4 = \delta_4 - \frac{1}{\epsilon\lambda}$. Introduce the following state-variables' transformation

$$z_1 = q_v \quad z_2 = \omega/\epsilon \quad z_3 = \epsilon\delta_v \quad z_4 = \epsilon\tilde{\delta}_4$$

with $\epsilon > 0$ so that system (51) is transformed into

$$\begin{aligned}
\dot{z}_1 &= \frac{\epsilon}{2}z_2 \\
J\dot{z}_2 &= -\epsilon\Gamma^i(t)\left(k_1 z_1 + \frac{1}{2}k_2\alpha\lambda(z_1 - \lambda z_3)\right) \\
\dot{z}_3 &= \epsilon\alpha(z_1 - \lambda z_3) \\
\dot{z}_4 &= -\epsilon\lambda z_4
\end{aligned} \tag{52}$$

and consider the so called "average system" of (52)

$$\begin{aligned}
\dot{z}_1 &= \frac{\epsilon}{2}z_2 \\
J\dot{z}_2 &= -\epsilon\Gamma^i_{av}\left(k_1 z_1 + \frac{1}{2}k_2\alpha\lambda(z_1 - \lambda z_3)\right) \\
\dot{z}_3 &= \epsilon\alpha(z_1 - \lambda z_3) \\
\dot{z}_4 &= -\epsilon\lambda z_4
\end{aligned} \tag{53}$$

where $\Gamma^i_{av}$ was defined in (20). Thus, proceeding in a fashion perfectly parallel to the one followed in the proof of Theorem 2 it can be shown that through an appropriate coordinate transformation system system (52) can be seen as a perturbation of system (53). Next, note that the correspondent of system (43) in the proof of Theorem 2 is given by

$$\begin{aligned}
\dot{w}_1 &= \frac{1}{2}w_2 \\
J\dot{w}_2 &= -\Gamma^i_{av}\left(k_1 w_1 + \frac{1}{2}k_2\alpha\lambda(w_1 - \lambda w_3)\right) \\
\dot{w}_3 &= \alpha(w_1 - \lambda w_3) \\
\dot{w}_4 &= -\lambda w_4
\end{aligned} \tag{54}$$

In order to prove robust exponential stability of (54) use the following Lyapunov function

$$V_o(w_1, w_2) = k_1 w_1^T\Gamma^i_{av}w_1 + \frac{1}{2}w_2^T J w_2 + \frac{1}{2}k_2\alpha\lambda(w_1 - \lambda w_3)^T\Gamma^i_{av}(w_1 - \lambda w_3) + \frac{1}{2}w_4^2 - 2\beta w_1^T J w_2$$

with $\beta > 0$. It is relatively simple to show that if $\beta$ is small enough, then there exist $0 < b_1 < b_2$ such that for all $J$ that satisfy Assumption 1 it holds that

$$b_1\|w\|^2 \leq V_o(w_1, w_2) \leq b_2\|w\|^2$$

Moreover, for all $J$ that satisfy Assumption 1 the following holds

$$\dot{V}_o(w_1, w_2) \leq -\beta J_{min}\|w_2\|^2 - \lambda w_4^2 \\ - [w_1^T w_3^T]\begin{bmatrix} (\Gamma_{av}^i)^{\frac{1}{2}} & 0 \\ 0 & (\Gamma_{av}^i)^{\frac{1}{2}} \end{bmatrix} M \begin{bmatrix} (\Gamma_{av}^i)^{\frac{1}{2}} & 0 \\ 0 & (\Gamma_{av}^i)^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} w_1 \\ w_3 \end{bmatrix}$$

where

$$M = \begin{bmatrix} [k_2\alpha^2\lambda^2 - \beta(2k_1 + k_2\alpha\lambda)]I & \beta k_2\alpha\lambda^2\left(\frac{1}{2} - \lambda\right)I \\ \beta k_2\alpha\lambda^2\left(\frac{1}{2} - \lambda\right)I & k_2\alpha^2\lambda^4 I \end{bmatrix}$$

Note that $M$ can be factored as follows (see Reference 15 p. 81)

$$M = N\begin{bmatrix} \left\{k_2\alpha^2\lambda^2 - \beta\left[(2k_1 + k_2\alpha\lambda) + \beta k_2\left(\frac{1}{2} - \lambda\right)^2\right]\right\}I & 0 \\ 0 & k_2\alpha^2\lambda^4 I \end{bmatrix}N^T$$

where

$$N = \begin{bmatrix} I & \frac{\beta}{\alpha\lambda^2}\left(\frac{1}{2} - \lambda\right)I \\ 0 & I \end{bmatrix}$$

Thus $M$ is positive definite for $\beta$ small enough, and system (54) is robustly exponentially stable.

The proof can be completed by using arguments similar to those used in the proof of Theorem 2. □

Considerations similar to Remarks 3 through 7 apply to the proposed output feedback; specifically, in practical applications gains $\alpha$ and $\lambda$ are often chosen by trial and error.

## SIMULATIONS

For simulation consider a satellite whose inertia matrix is equal to

$$J = \begin{bmatrix} 5 & -0.1 & -0.5 \\ -0.1 & 2 & 1 \\ -0.5 & 1 & 3.5 \end{bmatrix} \text{ kg m}^2 \tag{55}$$

The principal moments of inertia are then equal to $1.4947$, $3.7997$, and $5.2056$ kg m$^2$ (see Reference 16). The satellite follows a circular near polar orbit ($incl = 87°$) with orbit altitude of 450 km; the corresponding orbital rate is $n = 1.1\ 10^{-3}$ rad/s and the orbital period is about 5600 s . Without loss of generality the right ascension of the ascending node $\Omega$ is set equal to 0, whereas the initial phases $\alpha_0$ (see (10)) and $\phi_0$ (see (11)) have been randomy selected and set equal to $\alpha_0 = 4.54$ rad and $\phi_0 = 0.94$ rad.

First, check that for the considered orbit Assumption 2 is fulfilled. It was shown in Remark 1 that the assumption is satisfied if $\det(\Gamma_{av}^i) \neq 0$. In Fig. 1 the determinant of $1/T\int_0^T \Gamma^i(t)dt$ is plotted

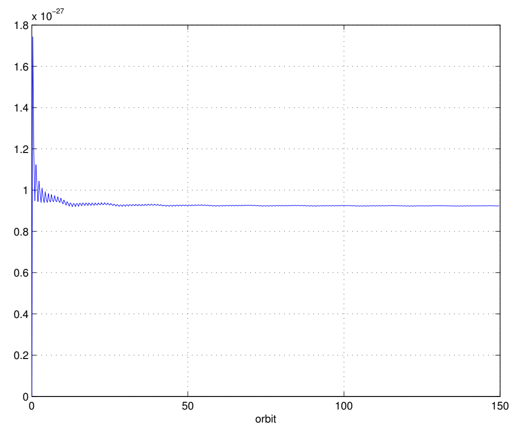



**Figure 1. Determinant of $1/T \int_0^T \Gamma^i(t)dt$.**

as function of $T$. The plot shows that $\det(1/T \int_0^T \Gamma^i(t)dt)$ converges to $9.23\ 10^{-28}$. It is of interest to compare the latter value with the value $9.49\ 10^{-28}$ determined using the analytical expression (21) which is valid when $\theta_m$ is approximated to $180°$.

Consider a rest-to-rest maneuver from the initial attitude

$$q(0) = [0.0343\ \ 0.1060\ \ 0.1436\ \ 0.9833]^T \tag{56}$$

with $\omega(0) = 0$, to the desired final attitude $(q, \omega) = (\bar{q}, 0)$. The initial attitude (56) corresponds to the following Euler angles

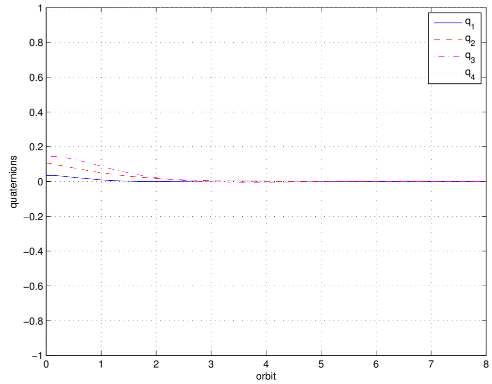
$$\phi(0) = 0.1 \text{ rad} \quad \theta(0) = 0.2 \text{ rad} \quad \psi(0) = 0.3 \text{ rad}$$

*State feedback.* For the considered rest-to-rest maneuver, the controller's parameters of the state feedback control (22) have been chosen by trial and error as follows $k_1 = 6\ 10^8$, $k_2 = 2\ 10^9$, $\epsilon = 10^{-3}$. Simulation's results are shown in Figure 2. Note that plots of the Euler angles are included even if redundant since plotting Euler angles is useful to clarify the spacecraft's motion. From Figure 2, we see that with those values for the controller's parameters, the spacecraft comes to rest at the desired attitude within 7 orbits, and the maximum dipole magnetic moment required is less than $3\ 10^{-3}$ A m$^2$. Moreover, there is a no overshoot except for a small one in the yaw angle. Thus, with the proposed PD-like control we have obtained performances very similar to those in Reference 16 where a so called Forward-Integrating Riccati-based control is emplyed; compared to the latter control, the PD-like control algorithm here proposed is simpler.

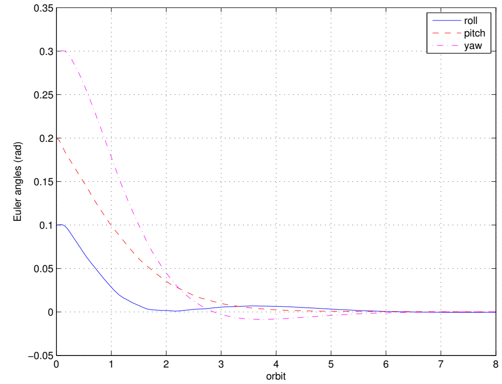
*State feedback with perturbed inertia matrix.* In order to test robustness of the designed state feedback with respect to uncertainties of the spacecraft's inertia matrix, new simulations of the same rest-to-rest maneuver were set up as follows. For the controller's parameters we used the previous values which were tuned for a spacecraft with inertia matrix indicated in (55); however, in the simulation we used the following perturbed value of the inertia matrix

$$J_{pert} = \text{diag}[1.4947\ 5.2056\ 3.7997] \text{ kg m}^2$$

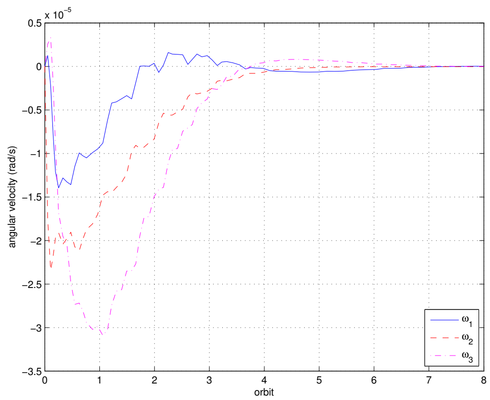
Note that the inertia matrix above is obtained if, for the spacecraft having inertia matrix $J$ as in (55), the body axis are changed and set equal to the principal inertia axis. Thus, since the smallest



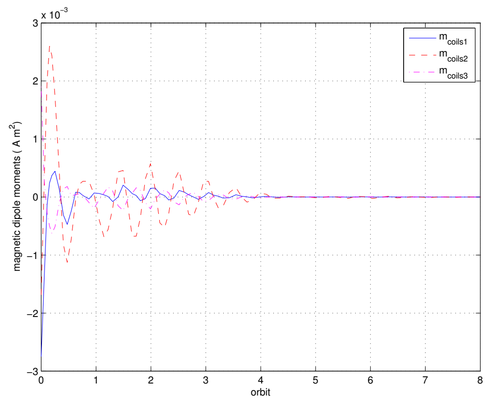

(a) Quaternions.




(b) Euler angles.




(c) Angular velocity.



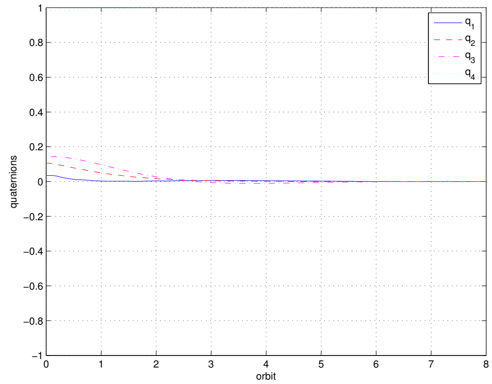

(d) Magnetic dipole moments.

**Figure 2. State feedback for the rest-to-rest maneuver.**

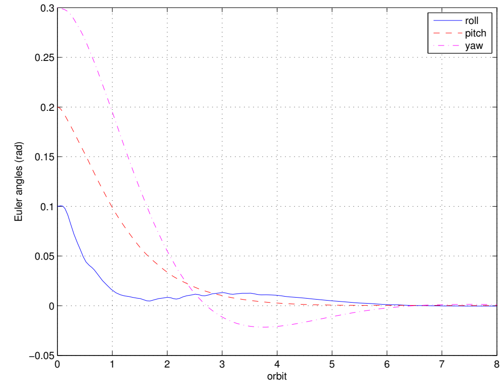
and largest principal moments of inertia of $J$ and $J_{pert}$ are identical, according to Theorem 2 (see Remark 3), local exponential stability will be preserved is spite of the perturbation on $J$ if the value $\epsilon = 10^{-3}$ previously chosen, is sufficiently small. The results of the simulation obtained using $J_{pert}$ are reported in Fig. 3. Plots show that exponential stability is preserved and performances are quite similar to ones with original inertia matrix; there is a slight increase in the overshoot of the yaw angle.

*State feedback with actuator saturation.* The capability of the the proposed state feedback to handle saturation in the magnetic actuators has been tested, too. This has been done implementing control law (48) with $m_{coils\ max} = 2\ 10^{-4}$ A m$^2$. For $J$ the original value (55) has been used. The corresponding time behaviors are plotted in Fig. 4. Note that the control system has become slower since the desired attitude is acquired within 12 orbits; moreover overshoot has become larger; both situations occurs also with control law employed in Reference 16.

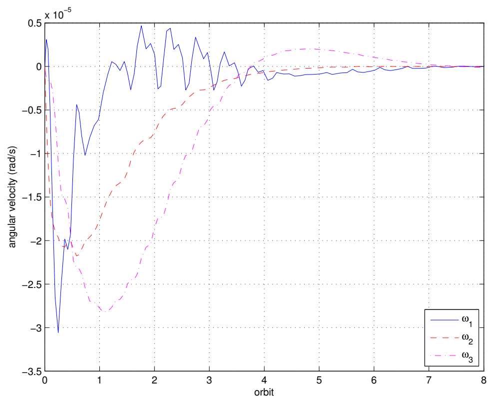
*Output feedback.* Consider the same rest-to-rest maneuver, and use output feedback (49) which has the advantage of not requiring measures of attitude rate. The values of controller's parameters have been determined by trial and error as follows $k_1 = 7\ 10^8$, $k_2 = 10^{10}$, $\epsilon = 10^{-3}$, $\alpha = 1\ 10^3$,



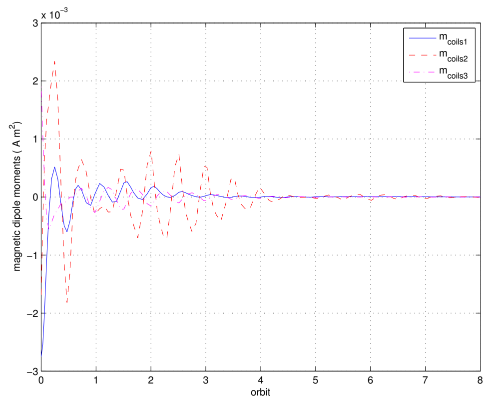

(a) Quaternions.




(b) Euler angles.




(c) Angular velocity.



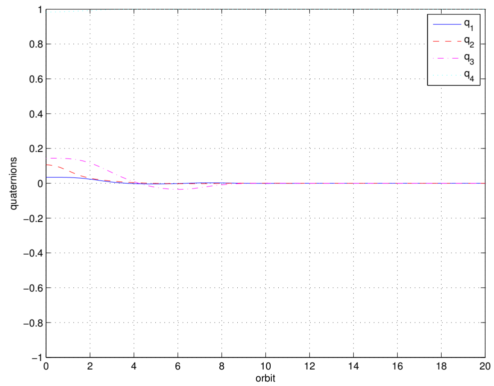

(d) Magnetic dipole moments.

**Figure 3. State feedback for the rest-to-rest maneuver with perturbed inertia matrix.**

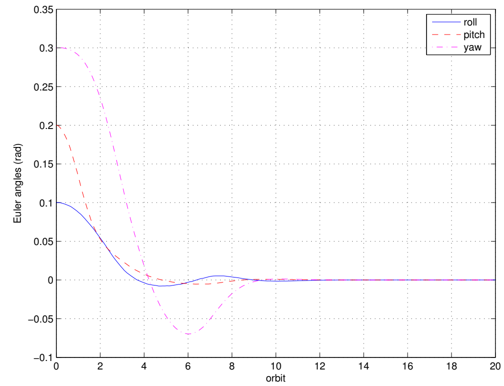
$\lambda = 1$. Simulation results are plotted in Fig. 5. Note that the spacecraft comes to rest at the desired attitude within 8 orbits as in the case of state feedback; however, the maximum magnetic dipole generated has increased to $4\ 10^{-3}$ A m$^2$.

*Output feedback with perturbed inertia matrix.* The robustness of the designed output feedback with respect to uncertainties of the spacecraft's inertia matrix has been tested in the same manner as with state feedback. The results of the simulation obtained using $J_{pert}$ are reported in Fig. 6. As in the case of state feedback, plots show that exponential stability is preserved and performances are quite similar to ones with original inertia matrix; there is a slight increase in the overshoot of the yaw angle.

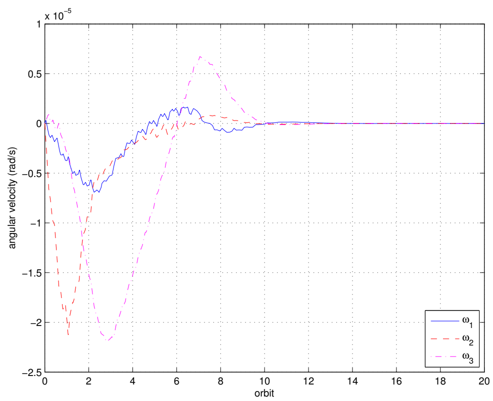
*Output feedback with actuator saturation.* Finally the performances of the proposed output feedback with saturation in the magnetic actuators have been evaluated, too. The corresponding time behaviors are omitted for lack of space. However, it turns out the the differences in the time behaviors with respect to the no saturation scenario, are similar to those detected when state feedback is employed.



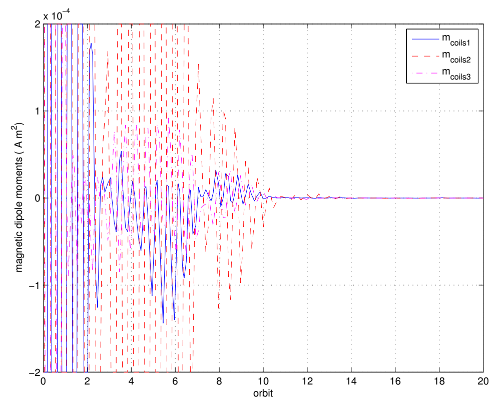

(a) Quaternions.




(b) Euler angles.




(c) Angular Velocity.




(d) Magnetic dipole moments.

**Figure 4. State feedback for the rest-to-rest maneuver with magnetic dipole moments saturation.**

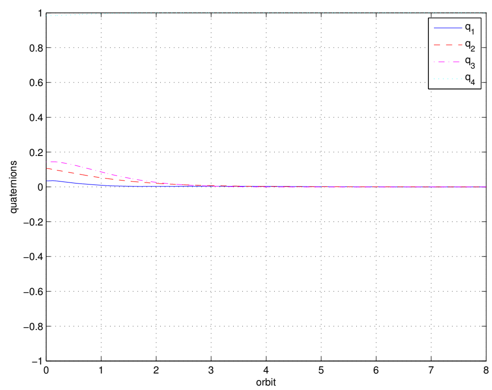
## CONCLUSIONS

Three-axis attitude controllers for inertial pointing spacecraft using only magnetorquers have been presented. An attitude plus attitude rate feedback and an attitude only feedback are proposed. With both feedback laws local exponential stability and robustness with respect to large inertia uncertainties are achieved. Simulation results have shown the effectiveness and feasibility of the proposed control designs

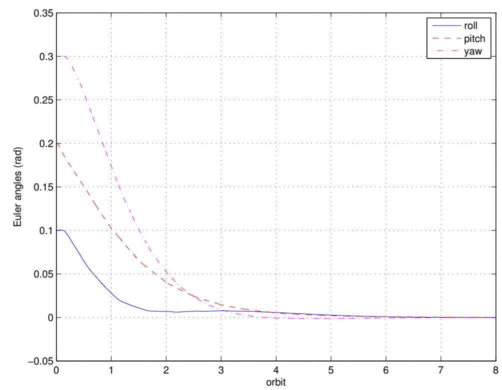
This work shows promising results for further research in the field; in particular, it would be interesting to extend the presented control algorithms to the case of Earth pointing pointing spacecraft; it would be also good to determine a control law which guarantees global convergence rather than just local.

## ACKNOWLEDGEMENT

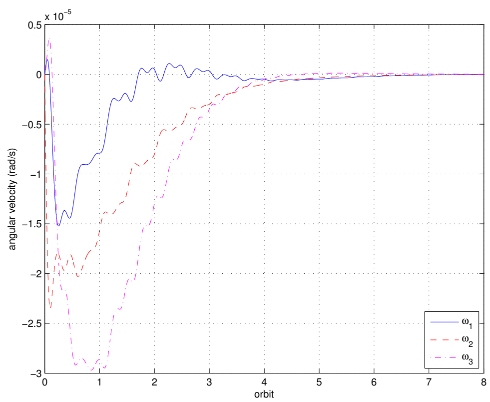
The author is grateful to Prof. P. Teofilatto for encouraging him to do research on magnetic control of spacecraft.



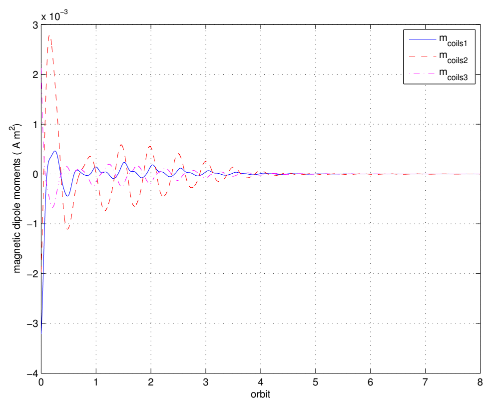

(a) Quaternions.




(b) Euler angles.

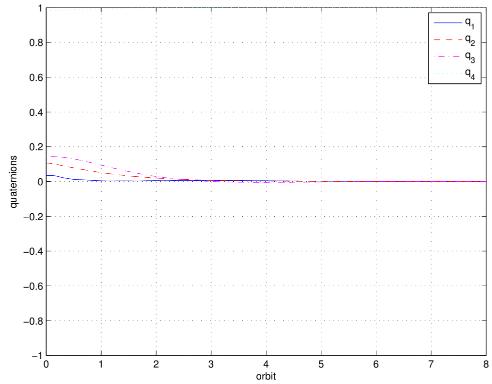



(c) Angular velocity.

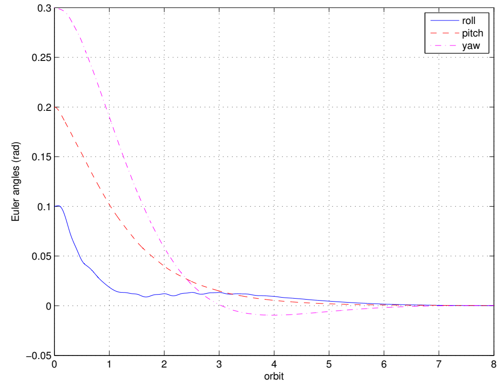



(d) Magnetic dipole moments.

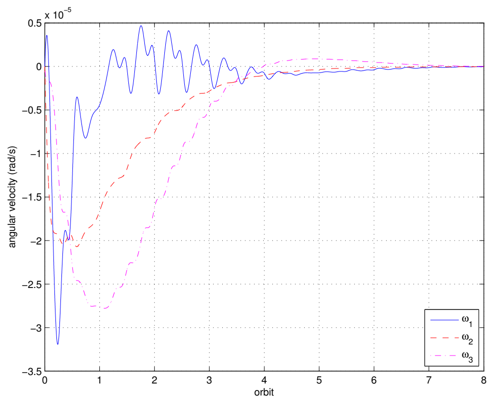
**Figure 5. Output feedback for the rest-to-rest maneuver.**



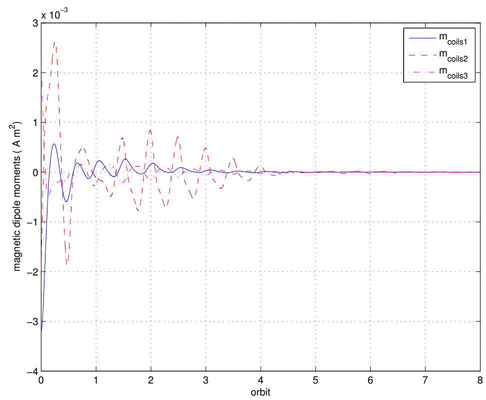

(a) Quaternions.




(b) Euler angles.




(c) Angular velocity.




(d) Magnetic dipole moments.

**Figure 6. Output feedback for the rest-to-rest maneuver with perturbed inertia matrix.**

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