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NATURAL INTERMEDIARIES AS ONBOARD ORBIT PROPAGATORS

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Short-term satellite onboard orbit propagation is required when GPS position measurements are unavailable due to an obstruction or a malfunction. In this paper, it is shown that natural intermediary orbits of the main problem provide a useful alternative for the implementation of short-term onboard orbit propagators instead of direct numerical integration. Among these intermediaries, Deprit's radial intermediary, obtained by the elimination of the parallax transformation, shows clear merits in terms of computational efficiency and accuracy. Indeed, this proposed analytical solution is free from elliptic integrals, as opposed to other intermediaries, thus speeding the evaluation of corresponding expressions. A comprehensive performance evaluation using Monte-Carlo simulations is performed for various orbital inclinations, showing that the analytical solution based on Deprit's radial intermediary outperforms a Dormand-Prince fixed-step Runge-Kutta integrator as the inclination grows.

INTRODUCTION

The main perturbation affecting low Earth orbit satellites is the Earth's oblateness, approximated mathematically by the presence of an additional term in the gravitational potential, which includes the second zonal harmonic J_2. The problem of the motion in this potential field is known as *the main problem* in artificial satellites theory, and was the subject of many studies over the years. Because the dynamics are nonintegrable (except for equatorial motion), and might even exhibit chaotic behaviour,[1,2] the only possible closed-form solutions may be made by obtained by approximating and/or averaging the full J_2 gravitational potential.

The most notable result in the context of the main problem was obtained by Brouwer.[3] By using the von Zeipel method,[4] integrable approximations of the J_2 potential were obtained. The result was later refined by several other authors, in order to eliminate singularities.[5,6] Most solution approaches were based on the Delaunay variables. Another approximation for the gravitational potential of an oblate spheroid is given by two fixed attraction centers, which is an integrable problem.[7] Vinti[8,9] used oblate spheroidal coordinates in order to find an approximating potential for zonal harmonics up to J_4.

In the presence of orbital perturbations, the most significant of which is J_2, as noted above – common onboard orbit propagators are planned to provide the satellite navigation

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solution for short time spans, usually a few orbits or less; for example, in the case of GPS outage or obstruction, onboard propagators would estimate the orbit in the absence of measurements. For these short time intervals, the accumulation of high-order effects is insignificant and can be neglected. In consequence, the model of forces to propagate may be notably simplified, further allowing for simple and robust fixed-step onboard numerical integration.

An alternative for onboard orbit propagation is the use of *intermediaries*. The aim of the intermediary is to capture in an integrable model the bulk of the dynamics of the original problem. Successful intermediaries must guarantee that the residual between the original problem and the intermediary is free from first-order secular effects. Also, it would be desirable that the intermediary may incorporate some second-order secular effects.

In artificial satellite theory, the short-term model of forces is commonly limited to the main problem, for which, as mentioned above, a variety of intermediaries have been proposed. However, hardware and software improvements allegedly made the performance of onboard numerical integration surpass the benefits of having an analytical solution. However, the benefit of analytical solutions is the inherent reduction of the phase space dimension, implying that fewer or no differential equations have to be integrated on board, which may reduce the computational time as well as memory allocation. This would be important for small satellites with limited computational abilities, such as Cubesats.

The effectiveness of *common* intermediaries, where integrability was found directly in the same variables as in the original problem,[10] was eventually improved when showing that all of them may be regrouped into the more general class of *natural* intermediaries.[11] For the latter, integrability is found after a contact transformation that converts the original problem into the intermediary, thus allowing for inclusion in the intermediary orbit of additional first-order short-periodic effects of the main problem.[12,13]

The purpose of the present research is to compare the efficiency of main problem intermediaries vis-à-vis direct numerical integration. Among these intermediaries, Deprit's radial intermediary, which is based on the elimination of the parallax transformation, shows real merits. Indeed, the fact that its analytical solution is free from elliptic integrals, as opposed to the Cid-Lahulla intermediary, renders the evaluation of corresponding expressions much faster. The performance of the radial intermediary is compared to a direct numerical integration of the main problem in Cartesian coordinates, and also to Brouwer's solution.[3] Extensive computations indicate that Deprit's radial intermediary may be a computationally-efficient alternative to direct numerical integration for implementation of short-term onboard orbit propagators.

THE MAIN PROBLEM HAMILTONIAN

The main problem Hamiltonian is given by

$$\mathcal{H} = -\frac{\mu}{2a} + \frac{\mu}{2r}\,\frac{\alpha^2}{r^2}\,C_{2,0}\left(1 - \frac{3}{2}\sin^2 i + \frac{3}{2}\sin^2 i\cos 2\theta\right), \tag{1}$$

where μ, α, and $C_{2,0} = -J_2$ are three physical parameters which define the gravity field,[*] standing for the earth's gravitational parameter, the earth's equatorial radius, and the second order zonal harmonic coefficient, respectively. The variable r is the radius from the earth's center of mass,

$$r = \frac{a\,(1-e^2)}{1+e\cos f}, \tag{2}$$

where f is the true anomaly, $\theta = f + \omega$ is the argument of latitude, and $(a, e, i, \Omega, \omega, M)$ are the semimajor axis, eccentricity, inclination, right ascension of the ascending node, argument of perigee, and mean anomaly, respectively, which constitute the modified classical set of orbital elements.[14] Besides, the eccentricity function

$$\eta = \sqrt{1-e^2}, \tag{3}$$

the *semilatus rectum*

$$p = a\,\eta^2, \tag{4}$$

and the mean motion

$$n = \sqrt{\mu/a^3}, \tag{5}$$

are commonly used, as well as the abbreviations $c \equiv \cos i$, $s \equiv \sin i$. In inertial earth-centered Cartesian coordinates, defined by the triad of unit vectors $\{\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z\}$, the Hamiltonian (1) yields the equations of motion

$$\ddot{X} = -\frac{\mu X}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{\alpha}{r}\right)^2\left(5\frac{Z^2}{r^2} - 1\right)\right] \tag{6a}$$

$$\ddot{Y} = -\frac{\mu Y}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{\alpha}{r}\right)^2\left(5\frac{Z^2}{r^2} - 1\right)\right] \tag{6b}$$

$$\ddot{Z} = -\frac{\mu Z}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{\alpha}{r}\right)^2\left(5\frac{Z^2}{r^2} - 3\right)\right] \tag{6c}$$

where the inertial position vector is $\boldsymbol{r} = (X,\, Y,\, Z)^T$ and the inertial velocity vector is $\boldsymbol{v} = \left(\dot{X},\, \dot{Y},\, \dot{Z}\right)^T$.

Because of the Hamiltonian formulation, each orbital element is assumed to be a function of a given set of canonical variables given by three coordinates and their respective conjugate momenta, as, for instance, Delaunay variables (ℓ, g, h, L, G, H):

$$\ell = M, \quad g = \omega, \quad h = \Omega, \qquad L = n\,a^2, \quad G = L\,\eta, \quad H = G\cos i, \tag{7}$$

or polar-nodal variables $(r, \theta, \nu, R, \Theta, N)$:

$$r = \frac{p}{1+e\cos f}, \quad \theta = f + \omega, \quad \nu = \Omega, \quad R = n\,a\frac{e}{\eta}\sin f, \quad \Theta = \sqrt{\mu\,p}, \quad N = \Theta\cos i. \tag{8}$$

[*] Alternatively, adequate units of length and time can be chosen to show that Eq. (1) depends only on $C_{2,0}$.

The inter-relations between the Cartesian coordinates and the polar-nodal variables can be found as follows. Define the unit vectors $\boldsymbol{n}_{1,2}$ as:

$$\boldsymbol{n}_1 = \begin{cases} \widehat{\boldsymbol{h} \times \boldsymbol{i}_z} & \boldsymbol{h} \times \boldsymbol{i}_z \neq \boldsymbol{0} \\ \boldsymbol{i}_x & \boldsymbol{h} \times \boldsymbol{i}_z = \boldsymbol{0} \end{cases} \; ; \; \boldsymbol{n}_2 = \hat{\boldsymbol{h}} \times \boldsymbol{n}_1. \tag{9}$$

Then

$$\begin{gathered} r = \|\boldsymbol{r}\| \, ; \qquad \begin{cases} \cos\theta = \hat{\boldsymbol{r}} \cdot \boldsymbol{n}_1 \\ \sin\theta = \hat{\boldsymbol{r}} \cdot \boldsymbol{n}_2 \end{cases} ; \qquad \begin{cases} \cos\nu = \boldsymbol{i}_x \cdot \boldsymbol{n}_1 \\ \sin\nu = \boldsymbol{i}_y \cdot \boldsymbol{n}_1 \end{cases} ; \\ R = \frac{\boldsymbol{r} \cdot \boldsymbol{v}}{r}; \qquad \Theta = \|\boldsymbol{r} \times \boldsymbol{v}\| \, ; \qquad N = (\boldsymbol{r} \times \boldsymbol{v}) \cdot \boldsymbol{i}_z. \end{gathered} \tag{10}$$

The inverse transformation is performed by:

$$\mathbf{R}_\nu = \begin{bmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \; \mathbf{R}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} ; \; \mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

$$\boldsymbol{r} = \mathbf{R}_\nu \, \mathbf{R}_i \, \mathbf{R}_\theta \left[\begin{array}{ccc} r & 0 & 0 \end{array} \right]^T, \; \boldsymbol{v} = \frac{\boldsymbol{h} \times \boldsymbol{r}}{r^2} + R \frac{\boldsymbol{r}}{r}, \tag{12}$$

where:

$$\cos i = \frac{N}{\Theta}; \;\; \sin i = \sqrt{1 - \left(\frac{N}{\Theta}\right)^2}; \;\; \boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} = \mathbf{R}_\nu \, \mathbf{R}_i \left[\begin{array}{ccc} 0 & 0 & \Theta \end{array} \right]^T. \tag{13}$$

In spite of the known general nonintegrability of the main problem,[1,2] approximate solutions to the dynamics determined by the Hamiltonian in Eq. (1) may be quite useful in practical applications. In particular, solutions by perturbation theory can be very accurate for small values of J_2.[15,16,17] Thus, the main problem Hamiltonian is written

$$\mathcal{H} = \mathcal{H}_{0,0} + \epsilon \, \mathcal{H}_{1,0} \tag{14}$$

where $\mathcal{H}_{0,0}$ represents the integrable dynamics, ϵ is a small parameter, and $\mathcal{H}_{1,0}$ is the perturbation Hamiltonian. The integrable part $\mathcal{H}_{0,0}$ is called an *intermediary* solution to the main problem, and the standard approach is to take it equal to the Keplerian Hamiltonian while all terms factored by J_2, which is of the order of 10^{-3} for the earth, are left in the perturbation,[3,18] namely,

$$\mathcal{H}_{0,0} = -\frac{\mu}{2a}, \tag{15}$$

$$\mathcal{H}_{1,0} = \frac{\mu}{r} \frac{\alpha^2}{r^2} \frac{1}{2} C_{2,0} \left(1 - \frac{3}{2} s^2 + \frac{3}{2} s^2 \cos 2\theta\right), \tag{16}$$

However, more sophisticated intermediaries have also been proposed. Thus, for instance, by simply splitting the main problem Hamiltonian into a *radial* part

$$\mathcal{H}_{0,0} = -\frac{\mu}{2a} + \frac{\mu}{2r} \frac{\alpha^2}{r^2} C_{2,0} \left(1 - \frac{3}{2} s^2\right), \tag{17}$$

and a *zonal* part

$$\mathcal{H}_{1,0} = \frac{\mu}{2r}\frac{\alpha^2}{r^2}C_{2,0}\,\frac{3}{2}s^2\cos 2\theta, \tag{18}$$

one arrives to the so-called Cid's intermediary in Eq. (17), whose integrability becomes apparent when the Keplerian part is expressed in polar-nodal variables

$$-\frac{\mu}{2a} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r}. \tag{19}$$

Indeed, Eq. (17) is of one degree of freedom in (r, R) and, therefore, integrable. Besides, it retains all the secular terms of the main problem up to the first order[13]*.

Even nowadays, intermediaries may be useful as approximate solutions to the main problem.[20] It is desirable that the intermediary be as accurate as possible; thus, in addition to having the same secular behavior as the main problem, at least up to $\mathcal{O}(J_2)$, it should incorporate as much as possible of the short-periodic terms. This gives rise to the *zonal* intermediaries. For instance, the main problem Hamiltonian can be reorganized as

$$\mathcal{H}_{0,0} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) + \frac{\mu}{2r}\frac{\alpha^2}{r^2}C_{2,0}\left(1 - \frac{3}{2}s^2 + \frac{3}{2}\frac{r}{p}\,s^2\cos 2\theta\right), \tag{20}$$

$$\mathcal{H}_{1,0} = \frac{\mu}{2r}\left(1 - \frac{r}{p}\right)\frac{\alpha^2}{r^2}C_{2,0}\,\frac{3}{2}s^2\cos 2\theta, \tag{21}$$

where the integrability of Eq. (20), historically the first main problem intermediary, which has the same degrees of freedom as the main problem, was proven by Sterne.[21]

While most common intermediaries[21,22,23] miss first-order short-periodic effects of J_2, a relevant one was proposed by Vinti,[8,9] whose intermediary is accurate up to second-order effects of J_2, and is also useful for computing a second-order solution of the main problem which is free from small divisors.[24]

DEPRIT'S RADIAL INTERMEDIARY

A radical approach was taken by Deprit: following Cid and Lahulla's proposal of computing the intermediary as a result of a canonical transformation,[12] Deprit introduced the concept of *natural* intermediaries, and showed that common intermediaries may be *naturalized* by computing the contact transformation that converts the main problem Hamiltonian into the corresponding intermediary.[11] In this way, the intermediary solution becomes accurate up to the first order of $C_{2,0}$ not only for secular and long-period terms, but it also accounts for short-period effects.

Based on the elimination of the parallax transformation, Deprit proposed his own intermediary which, as opposed to other intermediaries whose solution unavoidably depends

*Note that the equatorial Hamiltonian $\mathcal{H}_{\mathrm{E}} = -\frac{1}{2}(\mu/a)+\frac{1}{2}(\mu/r)\,(\alpha/r)^2\,C_{2,0}$, in spite of being a particular integrable case of the main problem,[19] does not comply with the traditional requirements on intermediary orbits because, as easily checked, its secular terms differ from those of the main problem in first order effects.

on elliptic functions, yields a closed-form solution in terms of trigonometric functions.[11] Since a faster evaluation of both the intermediary solution and the contact transformation is an essential prerequisite for the implementation of useful onboard analytical orbit propagators, we will reorganize Deprit's solution into a more convenient way for our purpose, as described next.

Ellimination of the parallax

The elimination of the parallax is a Lie transform[25,26] designed to simplify the main problem Hamiltonian by removing non-essential short periodic efects.[11] Based on the parallax identity

$$\frac{1}{r^m} = \frac{1}{r^2}\,\frac{1}{r^{m-2}} = \frac{1}{r^2}\left(\frac{1+e\cos f}{a\,\eta^2}\right)^{m-2}, \qquad m > 2. \tag{22}$$

Eq. (16) is rewritten as

$$\begin{aligned}\mathcal{H}_{1,0} \;=\; & \frac{\mu}{2a\,\eta^2}\,\frac{\alpha^2}{r^2}\,\frac{1}{2}C_{2,0}\Big[(2-3s^2)\,(1+e\cos f) \\ & +\frac{3}{2}s^2\,e\cos(f+2\omega)+3s^2\cos(2f+2\omega)+\frac{3}{2}s^2\,e\cos(3f+2\omega)\Big].\end{aligned} \tag{23}$$

Then, we find a canonical transformation

$$(\ell, g, h, L, G, H) \xrightarrow{\mathcal{T}} (\ell', g', h', L', G', H'),$$

which, up to the first order in J_2, transforms Eq. (23) into a new Hamiltonian $\mathcal{H}_{0,1}$ (in prime variables) that is chosen by removing the explicit appearance of the true anomaly from Eq. (23):

$$\mathcal{H}_{0,1} = \frac{\mu}{2p}\,\frac{\alpha^2}{r^2}\,C_{2,0}\left(1-\frac{3}{2}s^2\right). \tag{24}$$

In view of $c = N/\Theta$ and Eq. (4), it results from Eq. (24) that $\mathcal{H}_{0,1} \equiv \mathcal{H}_{0,1}(r, -, -, R, \Theta, N)$. That is, $\mathcal{H}_{0,1}$ is radial, as is the Keplerian Hamiltonian, cf. Eq. (19).

The first term W_1 of the generating function of the transformation is obtained by solving the *homological* equation

$$\{W_1, \mathcal{H}_{0,0}\} = \mathcal{H}_{1,0} - \mathcal{H}_{0,1}, \tag{25}$$

where the left-hand side of Eq. (25) stands for the Poisson bracket of W_1 and $\mathcal{H}_{0,0}$, which, in view of Eq. (15), depends only on the Delaunay action L. Namely, for any order m,

$$\{W_m, \mathcal{H}_{0,0}\} = n\,\frac{\partial W_m}{\partial \ell}. \tag{26}$$

Then, based on the differential relation between the true and mean anomalies

$$a^2\,\eta\,\mathrm{d}\ell = r^2\,\mathrm{d}f, \tag{27}$$

which is derived from the preservation of the angular momentum of Keplerian motion, W_1 is solved in closed form of the eccentricity by a simple quadrature. Namely,

$$W_1 = \frac{1}{n}\int (\mathcal{H}_{1,0} - \mathcal{H}_{0,1})\,\mathrm{d}\ell = \frac{1}{n}\int (\mathcal{H}_{1,0} - \mathcal{H}_{0,1})\,\frac{r^2}{a^2\,\eta}\,\mathrm{d}f,$$

which results in

$$\begin{aligned} W_1 &= n\,\alpha^2\,\frac{C_{2,0}}{8\eta^3}\Big[(4-6s^2)\,e\sin f \\ &\quad +3s^2\,e\sin(f+2\omega) + 3s^2\sin(2f+2\omega) + s^2\,e\sin(3f+2\omega)\Big]. \end{aligned} \tag{28}$$

Thus, in the polar-nodal set and up to the first order of $C_{2,0}$, the main problem Hamiltonian has been simplified to

$$\mathcal{T} : \mathcal{H} = \frac{1}{2}\left(R'^2 + \frac{\Theta'^2}{r'^2}\right) - \frac{\mu}{r'} + \frac{\Theta'^2}{2r'^2}\,\frac{\alpha^2}{p'^2}\,C_{2,0}\left(1 - \frac{3}{2}s'^2\right), \tag{29}$$

where $p' = \Theta'^2/\mu$ and $s'^2 = 1 - N'^2/\Theta'^2$. The first-order generating function is expressed in polar-nodal variables

$$W_1 = \frac{1}{2}C_{2,0}\,\alpha^2\frac{\Theta}{p^2}\left[\frac{p\,R}{\Theta}\left(1 - \frac{3}{2}s^2 - \frac{1}{2}s^2\cos 2\theta\right) - \left(1 - \frac{p}{r}\right)s^2\sin 2\theta\right] \tag{30}$$

from which the first-order corrections are derived:

$$\Delta r = p\left(1 - \frac{3}{2}s^2 - \frac{1}{2}s^2\cos 2\theta\right), \tag{31}$$

$$\Delta\theta = \left[\frac{3}{4} - \frac{5}{4}c^2 - \left(1 - 3c^2\right)\frac{p}{r}\right]\sin 2\theta + \frac{p\,R}{\Theta}\left[1 - 6c^2 + \left(1 - 2c^2\right)\cos 2\theta\right], \tag{32}$$

$$\Delta\nu = c\left[\left(\frac{1}{2} - 2\,\frac{p}{r}\right)\sin 2\theta + \frac{p\,R}{\Theta}\,(3 + \cos 2\theta)\right], \tag{33}$$

$$\Delta R = \frac{p\,\Theta}{r^2}\,s^2\sin 2\theta, \tag{34}$$

$$\Delta\Theta = \Theta\,s^2\left[\left(\frac{1}{2} - 2\,\frac{p}{r}\right)\cos 2\theta - \frac{p\,R}{\Theta}\sin 2\theta\right], \tag{35}$$

$$\Delta N = 0, \tag{36}$$

where $\Delta\xi = (\xi - \xi')/\kappa$, with

$$\kappa = \frac{1}{2}C_{2,0}\frac{\alpha^2}{p^2}, \tag{37}$$

the symbol ξ denotes any of the polar-nodal variables, $\xi \in (r, \theta, \nu, R, \Theta, N)$, and the right-hand side of each of Eqs. (31)–(36) as well as p in Eq. (37) should be expressed in prime variables when computing ξ, or in original ones when computing ξ'.

In summary, the Hamiltonian in Eq. (29) is the result of applying the transformation given by Eqs. (31)–(36) to the main problem Hamiltonian. Since Eq. (29) is cyclic in θ' and ν', it is *radial* and, therefore, integrable. Therefore, it is a natural intermediary solution to the main problem which is known as Deprit's radial intermediary (DRI).

It is worth noting that the elimination of the parallax does not need to rely on the algebra of the so-called parallactic functions, as it has been recently shown.[27] Hence Eqs. (31)–(36) are provided in a form that is slightly different from the original expressions given by Deprit (cf. p. 133 of Ref. 11). This simpler form has the added advantage of allowing for a faster evaluation, which is crucial to the present research.

For the shake of alleviating notation, in what follows we suppress the prime notation when there is no risk of confusion.

Intermediary solution

The analytical integration of the flow generated by DRI has been originally achieved by Deprit by means of a *torsion* transformation.[11] However, because the torsion provides the solution in an implicit form, its evaluation may require the use or inversion formulas or its alternative explicit construction by means of a new Lie transform. However, because the construction of an efficient solution to DRI which allows for a fast evaluation is the main goal of the present research, we find it more expedient to base our algorithm on the conventional approach to the solution of *quasi-Keplerian* systems.[11]

The DRI in Eq. (29) is written as

$$\mathcal{D} = \frac{1}{2}\left(R^2 + \frac{\tilde{\Theta}^2}{r^2}\right) - \frac{\mu}{r}, \tag{38}$$

where

$$\tilde{\Theta} = \Theta\,\sqrt{1-\varepsilon\,(2-6c^2)}, \qquad \varepsilon = \frac{1}{4}C_{2,0}\,\frac{\alpha^2}{p^2}, \tag{39}$$

and $\tilde{\Theta} \equiv \tilde{\Theta}(\Theta, N; \mu, \alpha, C_{2,0})$ plays the role of the angular momentum of the quasi-Keplerian system defined by Eq. (38).

Because N is an integral of the main problem and Θ is constant in the prime space, then $\tilde{\Theta}$ is also constant, and the solution to Eq. (38) is the usual ellipse centered at the focus. Indeed, the usual Hamilton-Jacobi reduction leads to the computation of the canonical transformation $(r, \theta, \nu, R, \Theta, N) \longrightarrow (\lambda, \gamma, h, \Lambda, \Gamma, H)$

$$(\lambda, \gamma, h, R, \Theta, N) = \frac{\partial \mathcal{S}}{\partial(\Lambda, \Gamma, H, r, \theta, \nu)} \tag{40}$$

where, because ν and θ are cyclic in Eq. (38), the generating function is chosen as

$$\mathcal{S} = \nu\,H + \theta\,\Gamma + S(r, \Lambda, \Gamma, H).$$

so that $\Gamma = \Theta$ and $H = N$. Besides,

$$\frac{1}{2}\left\{\left(\frac{\partial S}{\partial r}\right)^2 + \frac{\Gamma^2}{r^2}\left[1 - \varepsilon\left(2 - 6c^2\right)\right]\right\} - \frac{\mu}{r} = \Phi, \tag{41}$$

where now $p = \Gamma^2/\mu$, $c = H/\Gamma$, and $\Phi \equiv \Phi(-,-,-,\Lambda,\Gamma,H)$ has dimensions of energy.

Equation (41) can be solved by quadrature

$$S = \int \sqrt{Q}\,\mathrm{d}r, \tag{42}$$

with

$$Q = 2\Phi + 2(\mu/r) - (\tilde{\Theta}/r)^2. \tag{43}$$

where now $\tilde{\Theta} \equiv \tilde{\Theta}(\Gamma, H; \mu, \alpha, C_{2,0})$

Then, the transformation in Eq. (40) is written as

$$\lambda = \frac{\partial \Phi}{\partial \Lambda} I_1 \tag{44}$$

$$\gamma = \theta + \frac{\partial \Phi}{\partial \Gamma} I_1 - \left[1 + \varepsilon\left(2 - 12c^2\right)\right]\Gamma\, I_2 \tag{45}$$

$$h = \nu + \frac{\partial \Phi}{\partial H} I_1 - 6\varepsilon\, H\, I_2 \tag{46}$$

$$R = \sqrt{Q} \tag{47}$$

where the partial derivatives

$$\frac{\partial \varepsilon}{\partial H} = 0, \quad \frac{\partial \varepsilon}{\partial \Gamma} = -\frac{4}{\Gamma}\varepsilon, \qquad \frac{\partial \tilde{\Theta}^2}{\partial \Gamma} = 2\Gamma\left[1 + \varepsilon\left(2 - 12c^2\right)\right], \quad \frac{\partial \tilde{\Theta}^2}{\partial H} = 12\varepsilon\, H, \tag{48}$$

have been used, and

$$I_1 = \int_{r_{\min}}^{r} \frac{1}{\sqrt{Q}}\,\mathrm{d}r, \qquad I_2 = -\int_{r_{\min}}^{r} \frac{1}{\sqrt{Q}}\,\mathrm{d}\left(\frac{1}{r}\right). \tag{49}$$

The quadratures in Eq. (49) are solved by standard changes of variables. Thus, since $\Phi < 0$ for bounded motion, Eq. (43) can be written as

$$Q = \tilde{\Theta}^2\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r_2} - \frac{1}{r}\right) \tag{50}$$

where

$$r_{1,2} = \frac{(\tilde{\Theta}^2/\mu)}{1 \mp \sqrt{1 - (\tilde{\Theta}^2/\mu)\,(-2\Phi/\mu)}} = \frac{\tilde{p}}{1 \mp \tilde{e}} = \tilde{a}\,(1 \pm \tilde{e}) \tag{51}$$

and

$$\tilde{e}^2 = 1 - \frac{\tilde{p}}{\tilde{a}}, \quad (0 \le \tilde{e} \le 1), \qquad \tilde{p} = \frac{\tilde{\Theta}^2}{\mu}, \qquad \tilde{a} = -\frac{\mu}{2\Phi}. \tag{52}$$

Then, making

$$r = \tilde{a}\,(1 - \tilde{e}\cos u), \tag{53}$$

we find $\mathrm{d}r = \tilde{a}\,\tilde{e}\sin u\,\mathrm{d}u$, and hence

$$Q = \frac{\mu}{\tilde{a}}\,\frac{\tilde{e}^2\,\sin^2 u}{(1 - \tilde{e}\cos u)^2},$$

from which, taking into account that $r = r_{\min} \Rightarrow u = 0$, the first expression of Eq. (49) is easily solved to give

$$I_1 = \frac{1}{\sqrt{\mu/\tilde{a}^3}}\,(u - \tilde{e}\sin u) = \frac{\mu}{(-2\Phi)^{3/2}}\,(u - \tilde{e}\sin u) \tag{54}$$

which has dimensions of time.

On the other hand, making

$$r = \frac{\tilde{p}}{1 + \tilde{e}\cos v}, \tag{55}$$

we find $\mathrm{d}(1/r) = -(e/\tilde{p})\sin v\,\mathrm{d}v$, and hence

$$Q = \frac{\mu}{\tilde{p}}\,\tilde{e}^2\sin^2 v,$$

from which, taking into account that $r = r_{\min} \Rightarrow v = 0$, the last expression of Eq. (49) is trivially solved to give

$$I_2 = \frac{1}{\tilde{\Theta}}\,v, \tag{56}$$

which has dimensions of inverse of momentum. We note that the usual relation

$$\tan\frac{v}{2} = \sqrt{\frac{1+\tilde{e}}{1-\tilde{e}}}\tan\frac{u}{2} \tag{57}$$

holds. Finally, the traditional choice

$$\Phi = -\frac{\mu^2}{2\Lambda^2}, \tag{58}$$

provides the transformation in mixed variables

$$\lambda = u - \tilde{e}\sin u \tag{59}$$

$$\gamma = \theta - \left[1 + \varepsilon\left(2 - 12c^2\right)\right]\frac{\Gamma}{\tilde{\Theta}}\,v \tag{60}$$

$$h = \nu - 6\,\varepsilon\,\frac{H}{\tilde{\Theta}}\,v \tag{61}$$

$$R = \sqrt{\frac{\mu^2}{\Lambda^2} + 2\frac{\mu}{r} - \frac{\tilde{\Theta}^2}{r^2}} = \sqrt{\frac{\mu}{\tilde{a}}}\sqrt{\frac{\tilde{a}}{r}\left(2 - \frac{\tilde{p}}{r}\right) - 1} \tag{62}$$

$$\Theta = \Gamma \tag{63}$$

$$N = H \tag{64}$$

Alternatively, it is easy to check from Eq. (59), (53), and (55) that $\tilde{a}^2\,\tilde{\eta}\,\mathrm{d}\lambda = r^2\,\mathrm{d}v$, with the meaning of the preservation of the "angular momentum" $\tilde{\Theta}$ of the quasi-Keplerian motion. Then, Eq. (62) may be replaced by

$$R = \tilde{n}\,\tilde{a}\,\frac{\tilde{e}}{\sqrt{1-\tilde{e}^2}}\sin v = \sqrt{\mu}\sqrt{\frac{1}{\tilde{p}}-\frac{1}{\tilde{a}}}\sin v = \mu\sqrt{\frac{1}{\tilde{\Theta}^2}-\frac{1}{\Lambda^2}}\sin v. \tag{65}$$

where

$$\tilde{n} = \frac{\partial\Phi}{\partial\Lambda} = \frac{\mu^2}{\Lambda^3} = \sqrt{\frac{\mu}{\tilde{a}^3}}, \qquad \tilde{a} = \left(-\frac{\mu}{2\Phi}\right) = \frac{\Lambda^2}{\mu}, \tag{66}$$

The process described above is summarized in Algorithms 1 and 2.

Algorithm 1 $(\lambda,\gamma,h,\Lambda,\Gamma,H) \longrightarrow (r,\theta,\nu,R,\Theta,N)$

Evaluate N in Eq. (64), Θ in Eq. (63), and Φ in Eq. (58).
Make $p = \Theta^2/\mu$, $c = N/\Theta$, and evaluate $\tilde{\Theta}$ from Eq. (39).
Compute $\tilde{a}$, $\tilde{p}$, and $\tilde{e}$ from Eq. (52), and solve $u = u(\lambda,\tilde{e})$ from Eq. (59).
Evaluate r from Eq. (53), and compute v from Eq. (57)
Evaluate R in Eq. (62), with the same sign as $\sin v$; alternatively, evaluate Eq. (65).
Compute θ from Eq. (60), and ν from Eq. (61).

Algorithm 2 $(r,\theta,\nu,R,\Theta,N) \longrightarrow (\lambda,\gamma,h,\Lambda,\Gamma,H)$

Compute H from Eq. (64) and Γ from Eq. (63).
Evaluate $\mathcal{D}$ from Eq. (38) and make $\Phi = \mathcal{D}$.
Make $p = \Theta^2/\mu$, $c = N/\Theta$, and evaluate $\tilde{\Theta}$ in Eq. (39).
Compute $\tilde{a}$, $\tilde{p}$, and $\tilde{e}$ from Eq. (52)
Compute $\Lambda > 0$ from Eq. (58) or the second of Eq. (66).
Solve Eqs. (55) and (65) for v and compute u from Eq. (57)
Evaluate λ in Eq. (59), γ in Eq. (60), and h in Eq. (61)

SIMULATIONS

The performance of the DRI in terms of computational speed and accuracy is compared to the solution provided by a simple and reliable fixed-step fourth-order Runge-Kutta (R-K) integration of the Cartesian equations of motion (6). In particular, we use the Hairer-Wanner implementation* of the Dormand and Prince[28] method, which minimizes the error of the fifth-order solution and requires six function evaluations per integration step.[29]

Tests are performed in a set of low Earth near-circular orbits, with inclinations ranging from equatorial to polar, including the critical inclination. Specifically, we took the inclination values $i = 5, 50, 63.4$ and 89 degrees for the initial elements, while the other orbital elements were fixed to

$$a = 7000\,\mathrm{km}, \quad e = 0.005, \quad \omega = 10\,\mathrm{deg}, \quad \Omega = 0\,\mathrm{deg}, \quad f = 15\,\mathrm{deg}, \tag{67}$$

*The DOPRI5 code is publicly available at http://www.unige.ch/ hairer/prog/nonstiff/dopri5.f in all the cases tested.

The analytical solution based on DRI is, by definition, limited in accuracy because it only takes into account first order effects of $C_{2,0}$. For the orbits tested, this may mean a position error of the order of tens of meters at the end of one orbit. Conversely, the R-K solver allows for much higher accuracy depending on the step size chosen. This is illustrated in Fig. 1 in which the number of accurate digits in position and velocity is estimated as the decimal logarithm of the corresponding relative errors, which in turn have been computed as

$$\Delta = \frac{\|\boldsymbol{x} - \boldsymbol{\xi}\|}{\|\boldsymbol{\xi}\|}, \qquad \delta = \frac{\|\dot{\boldsymbol{x}} - \dot{\boldsymbol{\xi}}\|}{\|\dot{\boldsymbol{\xi}}\|},$$

where the "true" solution $(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$ is obtained with a higher-order, variable step size numerical integrator.

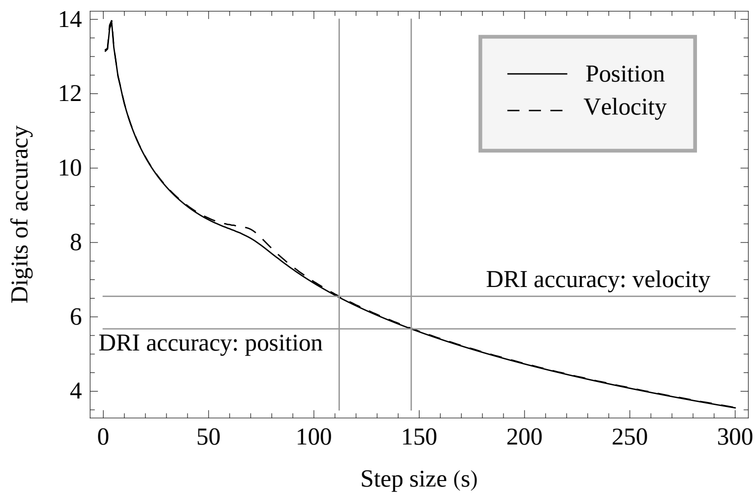

Figure 1. Digits preserved by the explicit R-K integration for different fixed step sizes. The horizontal lines at ~ 5.7 and 6.6 mark the accuracy obtained by the evaluation of DRI (initial conditions in Eq. (67) with $i = 55\,\mathrm{deg}$).

As shown in Fig. 1, which was constructed for a 90 minute run, corresponding to about one orbit of a typical test case, the best performance of the R-K solution is obtained with step sizes of about four seconds. The solution slightly deteriorates for smaller step sizes due to the accumulation of truncation errors, although this accumulation may be negligible for shorter integration intervals, say of 15 minutes. On the other hand, the R-K method used is unable to accurately reproduce the solution for higher step sizes, suffering the expected deterioration for increasing values of the step size. Similar curves were obtained for the other cases tested, as illustrated in Fig. 2.

For these short intervals, the accumulation of second order effects is barely noted in the evaluation of DRI, whose accuracy remains constant to all practical effects when compared to the true solution. Thus, for instance, for the 55 deg inclination orbit the relative errors in Cartesian coordinates are of about 2×10^{-6} in position and 3×10^{-7} in velocity, corresponding to the horizontal gray lines in Fig. 1. As a result, roughly speaking we may say

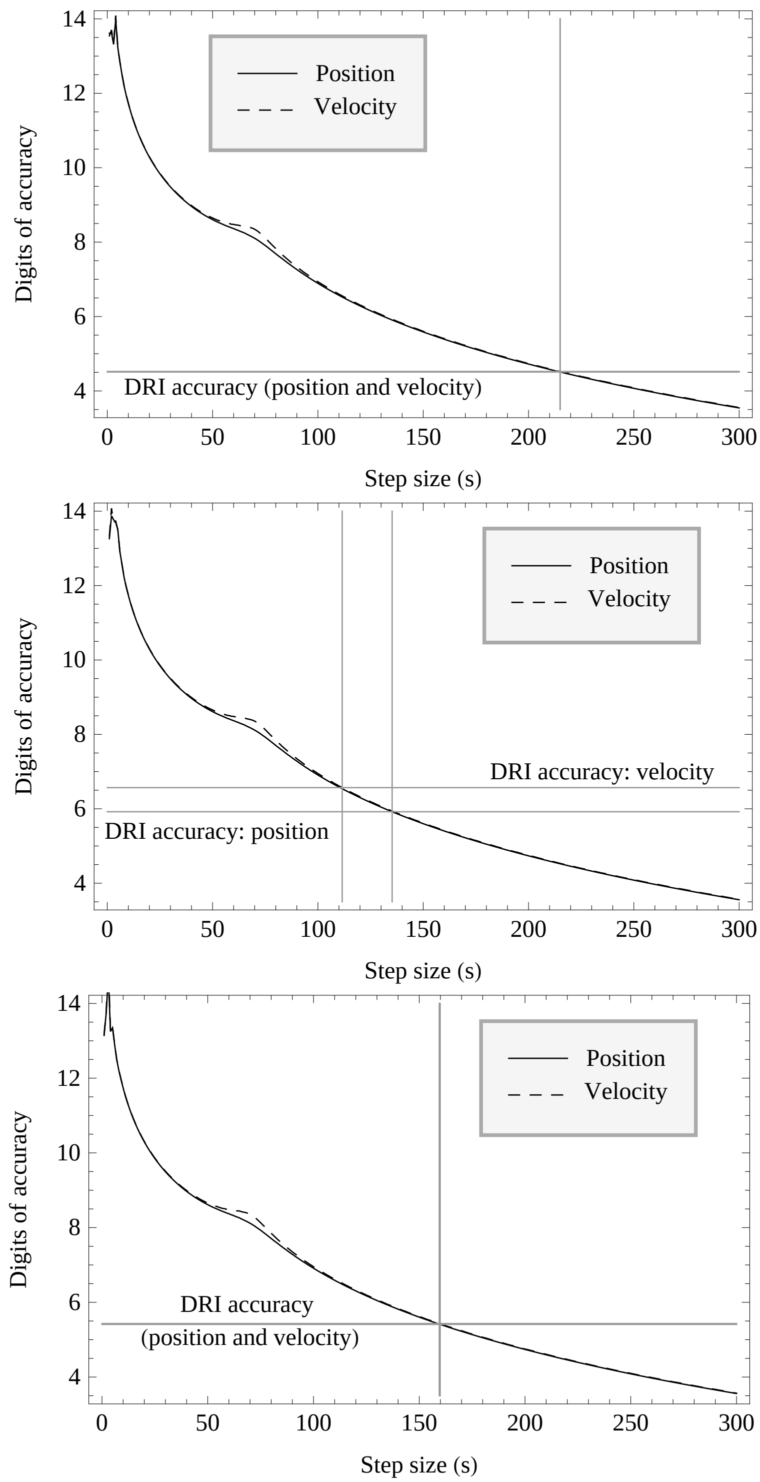

Figure 2. Digits preserved by the explicit R-K integration for different fixed step sizes. Initial conditions in Eq. (67) with, from top to bottom, $i = 5$, 63.4 and $89\,\mathrm{deg}$.

that for step sizes shorter than 2 minutes the fixed step R-K solution is more accurate than DRI, for step sizes between 2 and 2.5 minutes both methods show analogous performance, whereas the performance of DRI clearly exceeds this of the R-K integration for larger step sizes. Similar figures were obtained for the other cases tested, except for the expected deterioration of the analytical solution for the lower inclination orbits, a case in which the effects of the $C_{2,0}$ perturbation are more important. In these cases, one additional digit of accuracy can be lost by DRI analytical solution, as it may clearly be observed in the top plot of Fig. 2.

However, in spite of the apparent advantages of the R-K over the DRI for normal step sizes, this is not the case under discussion, because the GPS provides the initial state of the satellite to be onboard propagated with some uncertainty, which may be of several meters in position and several centimeters per second in velocity. Hence, one may check that the DRI and R-K propagations result in analogous statistics even in the case in which the highest precision R-K integration is used. Indeed, we carried out 1000 Monte-Carlo simulations for a 1000 seconds run, assuming that uncertainties in position and velocity fit both to a normal distribution. In particular, we assumed a Gaussian noise of zero mean and 5 meters of standard deviation (STD) for each coordinate, and zero mean and 2 cm/s STD for each velocity component, both for DRI and the R-K integration with a constant step size. The actual orbit is assumed to be given by a precise numerical integration which assumes no errors in the initial conditions, and the errors of each propagation are computed each second. Brouwer's solution was also included in the simulations using Deprit and Rom's variant of Lyddane's non-singular formulation,[15,5] for the case of a pure first order solution (BL1), but also including second order secular terms (BL2).

Results for the different cases tested are summarized in Table 1, where the mentioned loss of accuracy for low inclination orbits may be also noted for DRI, but also for BL1 and BL2 for the same reasons. Propagation of orbits close to the critical inclination with an analytical solution does not make sense, although some patches may be introduced in these kinds of theories to deal properly with critically inclined orbits,[30,31] so we do not provide provide BL1 and BL2 results for that case in Table 1.

The computations were performed with a 2.3 GHz Intel Core i7 Processor and 8 GB of RAM. The algorithm was coded in Fortran 77 and compiled in 64-bit with the Absoft compiler for the Leopard operating system. Under these conditions, a single R-K evaluation averages to ~ 460 nanoseconds, whereas the evaluation of DRI solution lasts ~ 385 nanoseconds, about 20% faster. Form our trials we note that the evaluation speed may change slightly depending on compiling options, but the balance is always favorable to DRI. Evaluation of Brouwer's analytical solution, even in the case of BL1, is much more computationally expensive, and more than doubles the DRI figures in any of the cases tested.

Inclination (deg):	5	55	63.4	89	Method
Mean error in $\boldsymbol{x}$ (m):	18.7025	20.0818	19.4419	18.3899	R-K
	20.5101	20.2656	19.5721	19.2209	DRI
	45.5822	22.0718	—	28.8093	BL1
	30.3993	21.6227	—	26.6436	BL2
STD in $\boldsymbol{x}$ (m):	9.8004	11.0657	10.5404	9.6432	R-K
	10.8558	11.0503	10.4815	9.3659	DRI
	11.6365	11.9103	—	9.7773	BL1
	9.4604	11.7066	—	9.0531	BL2
Mean error in $\dot{\boldsymbol{x}}$ (cm/s):	3.7226	4.0852	3.9155	3.6913	R-K
	3.9553	4.0878	3.9185	3.7074	DRI
	6.1881	4.3799	—	4.3271	BL1
	4.7645	4.3221	—	4.1377	BL2
STD in $\dot{\boldsymbol{x}}$ (cm/s):	0.8490	1.1645	1.0827	0.7646	R-K
	1.0971	1.1653	1.0840	0.7751	DRI
	1.7617	1.3592	—	1.0950	BL1
	0.8392	1.3036	—	0.9993	BL2

Table 1. Summary of statistics.

CONCLUSIONS

A careful rearrangement of the involved formulae in the solution of Deprit's radial intermediary, which provides analytical approximation to the main problem solution, converts it into a real alternative to the usual Runge-Kutta integration used by onboard orbit propagators. Thus, on the one hand it provides improved performance in terms of the computational burden. On the other hand, in spite of its inherent limitations to the first order of J_2 against the higher precision of the fixed-step Runge-Kutta integration for small step sizes, both methods share similar accuracy statistics due to the uncertainties with which the satellite state is known. Finally, the fact that analytical solutions are free from a step by step evaluation, makes Deprit's radial intermediary much more versatile than its Runge-Kutta competitor.

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