THE ROLE OF DYNAMICAL MODELS IN BALLISTIC CAPTURE: 
THE PERTURBED SUN-PLANET CASE

Zong-Fu Luo, Francesco Topputo, Franco Bernelli, and Guo-Jian Tang

The role of the dynamical models plays in the ballistic capture orbits is analyzed. The ballistic capture problem has been studied in the planar restricted three- and four-body problems among the previous works. This paper extends it to the spatial space and the real gravitational field in the solar system. Three dynamical models, including CR3BP, ER3BP, and ephemeris, are introduced to study the influence of these models. The spatial stability is defined by a semi-plane in an inertial frame. The initial conditions are itemized into four types: stable, unstable, crash, and acrobatic. The ballistic capture orbits are generated around the Mercury, Venus, Mars, Jupiter, and Saturn, where the major gravities root from the sun and the planets themselves. The ballistic capture transfers are analyzed in the planar and spatial conditions.

INTRODUCTION

A spacecraft may be ballistically captured into a stable orbit around the target celestial body without any fuel consumption under the help of the multi-body dynamics, except the conventional propulsive braking maneuver. Obviously, the precondition for the ballistic capture orbit is that the spacecraft is governed by at least two gravitational forces during the approach to the target body. The more complex the dynamics is, the efficient transfers might be derived. This is a completely new concept from the classical Keplerian dynamics, and also labeled as low-energy transfer. The first application of the low-energy transfer was proposed to rescue the “broken-down” Japanese spacecraft Hiten at the expense of additional flight time in 1991. Thereafter, the ballistic capture orbits at arrival have been considered for the lunar exploration SMART-1 (ESA), the recent mission GRAIL (NASA) to send two twinborn spacecraft into lunar orbit, and the forthcoming Mercury mission BepiColombo (ESA) and European Student Moon Orbiter (ESMO, ESA).

Extensive analyses show that the low-energy transfers could allow us to save 10%-20% propellant than the patched-conic transfers, which is extremely meaningful for the lunar or interplanetary missions. However, this requires more preanalyses and considers strict constraints before receiving a satisfactory solution, even massive optimization works. As is known that the general three-body and n-body problems (n > 3) are still opening for us other than the two-body one. There are only five constant-pattern solutions found in three-body problem so far, namely the five Lagrangian points $L_1$ to $L_5$. Consequently, numerical methods are feasible instruments to study the ballistic capture activities and the long-term behavior of the spacecraft when two or more attractions simultaneously

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act on it. For simplicity, the mass of the spacecraft is deemed to be significantly small than that of the bodies and so that it does not influence the motion of the gravitational system, and these systems are defined as “restricted”. Note that numerous works reported in literature investigate the ballistic capture transfers with the planar, circular and elliptical restricted three-body problem models (PCR3BP and PER3BP), although they are simplified and nonautonomous. The difference between the above two dynamical models locates on whether the eccentricity or true anomaly of the target body with respect to the major perturbed body is considered or not. In fact, the ballistic capture orbits are susceptible to the relative distance between the primaries. Circi and Teofilatto confirmed that the capture is easier when the planet locates at its perihelion. Hyeraci and Topputo studied the role of the true anomaly and indicated that the efficient ballistic capture happens when the spacecraft approaches the target body with an opened zero velocity surface (ZVS) and closed after it is “trapped” (Figure 1(a)). Although the planar three-body problem provides a convenient framework in which to understand the dynamical process of the ballistic capture, the real application requires a spatial transfer trajectory and assesses in a full ephemeris model. Romagnoli and Circi established a three-dimensional geometric framework for the low-energy transfer trajectories, with a plane rotated by two angles around the line of Earth-moon direction in the synodic frame. In Reference 20, a similar definition is presented to extend the planar ballistic capture orbits to spatial cases.

In this analysis, the ballistic capture transfers are analyzed in the sun–planet system, that is the models in which the spacecraft trajectory about a planet is mostly influenced by the sun, and no moons are considered. The approach and revolution geometry of the trajectories in three dimensions is studied, with a pre-defined semi-plane at the initial epoch. As mentioned above, the ballistic capture dynamics was mainly analyzed in simplified models, CR3BP and ER3BP. In this work, the three-dimensional, full-ephemeris, n-body model is considered as well. The focus is on assessing to what extent useful information are pruned away in the simplified models (circular and elliptic three-body models) with respect to the real system. In this perspective, the role played by the dynamical models is analyzed. Not only, as the influence of the model parameters is also accounted for. This is done by studying the ballistic capture dynamics into a number of sun-planet systems, where the mass ratios vary significantly. The purpose of this investigation is to infer insights useful for the preliminary trajectory design in a multi-body environment.
Theoretically speaking, the ballistic capture may occur in any \( n \)-body models \((n \geq 3)\), such as the typical dynamical system in the solar system: the sun–planet and the sun–planet–moon cases.\(^2\,^3\) Only the former situation is considered in this work. As mentioned above, the PCR3BP and PER3BP models are conventional simplifications to approximate the real gravitational environment. Then, the motion of the massless object is governed by two mutually interacting particles revolving around each other in a circular or elliptical orbit about their barycenter. The orbits of three masses are further assumed to all lie in a common plane instead of three-dimensions, and the problem is degenerated as the “planar” problem.\(^8\) These models have found many applications in astrodynamics; indeed, it gives a good approximation to the real dynamics around a planet, especially for the case when the eccentricity of the planet is taken into account. However, this paper preferably focuses on the influence arisen by the spatial property and the performance using different dynamical models during the trajectory propagation, for instance, CR3BP, ER3BP, and ephemeris models.

Obviously, the sun-Earth-moon or sun-Jupiter-Galilean moons or sun-Saturn-Titan systems are typical \( n \)-body problems \((n > 3)\) in the solar system and therefore are not considered in this paper. These distant celestial bodies, i.e., Uranus, Neptune, and Pluto–Charon, are also eliminated from the consideration of ballistic capture transfers. Thus, the attentions are paid to the following five cases: the sun–Mercury, sun–Venus, sun–Mars, sun–Jupiter, and sun–Saturn systems. For convenience sake, the major perturbed body sun as “primary” body and the capture planet is termed as “secondary” body in the following context.

Reference Frames

In Reference 12, Hyaraci and Topputo pointed out it might be more appropriate and accurate to view the behavior of the trajectory in the planet-centered inertial frame. Moreover, a rotating frame can help us to analyze and understand how the gravity forces capture the massless object without any control maneuvers, whether temporarily or permanently. For this reason, the integration is performed in an inertial frame: EME2000, which represents the Earth Mean Equator and Equinox at the epoch J2000. This reference frame is also used in the JPL Planetary and Lunar Ephemeris DE405, not exactly coincident, but with a slight rotation under 0.1 arcsecond, which is ignored in this paper.\(^21\) This reference frame will demonstrates its advantage when considering the ballistic capture in ephemeris model. In practice, the center of the EME2000 coordinate system is located at the center of the secondary, instead of the Earth’s center.

In order to analysis the influence to the ballistic capture from the sun, a body central radial-tangential-normal (RTN) frame is introduced, whose x-axis is defined from the sun to the prescribed planet, z-axis comes from the normal of the planet around the sun, and y-axis completes the right-handed triad, as the frame \(x_r y_r z_r\) shown in Figure 2(a). In fact, this frame is an inertial frame and fixed at where the RTN frame stands at the initial epoch \((t_0)\). For this reason, the reference frame is time-dependent and denoted as RTN@Epoch for short. A transformation matrix between EME2000 and RTN@Epoch can be obtained from the state of the planet relative to the sun at \(t_0\).

For scientific application and mission management, these parameters with respect to the body central frame, for example, the inclination and the solar elevation angle, are more visible and understandable in comparison to EME2000 and RTN@Epoch. Thus, another one inertial frame composed by the spin-axis and the equator of the planet at a specific epoch is obtained, titled as body mean
Equator frame (denoted as BME@Epoch for simplicity’s sake, see $x_b y_b z_b$ in Figure 2(a)). More detailed information on the definition and transformation can be found by referring to Reference 21.

Besides the above three inertial frames, a barycentric rotating frame (BRF) is addressed for further analyses about the ballistic capture trajectories, as illustrated in Figure 2(b). To be more specific, the approach direction, i.e., from interior (traversing the bottle-neck near $L_1$) or exterior (traversing the bottle-neck near $L_2$), and Jacobian constant (CR3BP) can be explicitly given in this frame. Similar to the CR3BP, the BRF is further inducted into the ER3BP and ephemeris models. The distance between the primary and secondary is time-variable in ER3BP, and can be transformed to an isotropically pulsating coordinate, in which the dimensionless length is time-dependent and the Lagrange points are positioned in the same place as in the CR3BP frame. The situation in the ephemeris model is more intricate than the ER3BP framework, which is beyond the scope of this paper. Therefore, the rotating frame is labeled as BRF@Epoch for short.

Circular Restricted Three-body Problem

The equation of motion for a spacecraft in the CR3BP model with respect to an inertial frame is given by

$$\ddot{r} = -\frac{\mu_p}{r^3} r - \mu_s \left( \frac{r - r_s}{|r - r_s|^3} + \frac{r_s}{r_s^3} \right)$$

where $r$ and $r_s$ are the inertial positions of the spacecraft and the sun relative to the planet, respectively, and $\mu_s$ and $\mu_p$ are the gravitational parameters of the sun and the planet.

The equation of motion is also formulated in the rotating frame, in which the positions of the primary and secondary are fixed at $(-\mu, 0, 0)$ and $(1 - \mu, 0, 0)$, as well as the two Lagrange points $L_1$ and $L_2$ used in this study. The mass parameter $\mu$ is defined by $\mu = \frac{m_2}{m_1 + m_2}$, where $m_1$ and $m_2$ are the masses of the primary and secondary, respectively. The distance between the two gravitational bodies, the angular velocity, and the gravitational constant are all unity. Thus, the equation in the rotating frame is written as

$$\ddot{r} + 2 \begin{bmatrix} -\dot{y} \\ \dot{x} \\ 0 \end{bmatrix} = \frac{\partial \Omega}{\partial \mathbf{r}}$$

Figure 2. Geometry of the reference frames. The boundary of the forbidden region is ZVS.
where the position vector \( \mathbf{r} = [x, y, z]^T \) and the potential function is

\[
\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1 - \mu)
\]  

and \( r_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}, r_2 = \sqrt{(x + \mu - 1)^2 + y^2 + z^2} \). A classic Jacobi integral exists in the CR3BP model, as

\[
C = 2\Omega - v^2
\]

where \( v = \sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)} \) is the velocity of the particle in the rotating frame. In principles, the value \( v^2 < 0 \) in Eq. (4) imposes an forbidden region for the spacecraft with a given Jacobi constant, and \( v = 0 \) in Eq. (4) defines the ZVS, which delimits the boundary of the allowable motion (see Figure 2(b)). As explained previously, it is rather convenient to transform the state from this rotating frame to inertial frame.8,22 Note that the component in the z-axis is considered for the analyses of the out-of-plane behavior, instead of the “planar” CR3BP model.

**Elliptical Restricted Three-body Problem**

The Eq. (1) with respect to the inertial frame is still available in the ER3BP model, and the only difference is the calculation of the distance \( r_s \). It is assumed that the motion of the spacecraft moves under the gravitational field produced by the mutual “elliptic” motion of the primary and secondary, rather than the circular orbit. The CR3BP model can be thought as a nontrivial solution of the ER3BP with zero eccentricity to a certain extent.

From the view of the rotating frame, the ER3BP model can be transformed into a homologous description as the Eq. (2). This new framework isotropically pulsates as the distance between the primary and secondary, which is assumed to be the length unit. In this case, the equilibrium points locate at exactly the same position as the CR3BP. A time-dependent or anomaly-dependent integral can be obtained from this modified model, but not constant as the Jacobi integral in CR3BP. More details about the definition and transformation to the inertial frame can be found in Reference 22,23.

**Ephemeris Model**

The CR3BP and ER3BP provide a sufficiently accurate approximation to the real-world trajectories for the primary design of ballistic capture transfers in the sun–planet system, the designers will consider the problem in a full ephemeris model eventually, and evaluate the influence of the dynamical simplifications. In this model, the real-time parameters of the sun and the planet is retrieved from a specific ephemeris data, such as JPL Planetary and Lunar Ephemeris DE405. The reader is recommended to refer to Reference 24 for more details about DE405. The perturbations from other bodies in the solar system are taken into account in addition to the sun and the target planet. The equation of motion involving point masses is rewritten as

\[
\dot{\mathbf{r}} = -\frac{\mu_p}{r^3}\mathbf{r} - \mu_s \left( \frac{\mathbf{r} - \mathbf{r}_s}{|\mathbf{r} - \mathbf{r}_s|^3} + \frac{\mathbf{r}_s}{r_s^3} \right) - \sum_{i=1}^{n} \mu_i \left( \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} + \frac{\mathbf{r}_i}{r_i^3} \right)
\]

where \( r_i \ (i = 1, 2, ..., n) \) are the inertial positions of the perturbed bodies relative to the planet, and \( \mu_i \) are their gravitational parameters. Although these perturbations are small quantities in comparison to the gravitational accelerations from the primary and secondary, it may sometimes affect the solutions of the low-energy transfers due to its high nonlinearities. From this point of view,
the dynamics with full ephemeris can also be comprehended as a “perturbed restricted three-body
problem”.

A time-variable rotating frame is defined in the full ephemeris model, whose transformation
relationship to the inertial frame is beyond the scope of this paper and omitted here Reference 22,23.

**Numerical Method**

The propagations of the ballistic capture transfers in this analyses were implemented by a 7th/8th
order Runge–Kutta–Felhberg integrator with an automatic step-size control and an integration toler-
ance of $10^{-12}$. The computations in the CR3BP, ER3BP, and full ephemeris models were performed
using a Matlab codes under a mutual program structure. Two approaches were used to generate the
position and velocity of the required bodies, one is entirely defined by the input parameters, i.e., the
orbital elements of the bodies with respect to a special center, one uses the aforementioned DE405
to produce the current state of the bodies in the solar system. The former method is for the cal-
culation of the capture sets in the CR3BP and ER3BP models, and the latter is for the ephemeris
model.

**PARAMETER SETTINGS AND PERFORMANCE INDICES**

**Simulation Parameters**

As mentioned in the previous section, five planets are considered in this paper, including the
Mercury, Venus, Mars, Jupiter, and Saturn. Table 1 gives the relevant parameters about the five
planets, such as the orbital parameters, period, radius, and sphere-of-influence (SOI). The term $\mu$
is the mass ratio between the planet and the sun. The values $e_p$ and Period are the mean eccentricity
and period of the planet moving around the sun. The parameter Radius is the mean radius of each planet. The inclination $i_{pb}$ is the angle between the body equator and the orbital plane of the planet around the sun. Note that the angle $i_{pb}$ of the Venus is larger than 90 deg
due to its retrograde self-rotation. The last column of this table represent the SOI of the planet with
respect to the sun, which is defined as $\rho_p (m_2/m_1)^{\frac{2}{3}}$, where $\rho_p$ is mean distance between the planet
and the sun. 24

<table>
<thead>
<tr>
<th>Planet</th>
<th>$\mu_p$, km$^3$/s$^2$</th>
<th>$\mu$</th>
<th>$e_p$</th>
<th>Period, year</th>
<th>Radius, km</th>
<th>$i_{pb}$, deg</th>
<th>SOI Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>2.203E+04</td>
<td>1.660E-07</td>
<td>2.056E-01</td>
<td>0.24</td>
<td>2.439E+03</td>
<td>0.035</td>
<td>45.92</td>
</tr>
<tr>
<td>Venus</td>
<td>3.249E+05</td>
<td>2.448E-06</td>
<td>6.774E-03</td>
<td>0.62</td>
<td>6.051E+03</td>
<td>177.36</td>
<td>101.80</td>
</tr>
<tr>
<td>Jupiter</td>
<td>1.267E+08</td>
<td>9.539E-04</td>
<td>4.728E-02</td>
<td>11.86</td>
<td>7.149E+04</td>
<td>3.12</td>
<td>674.20</td>
</tr>
<tr>
<td>Saturn</td>
<td>3.794E+07</td>
<td>2.858E-04</td>
<td>5.379E-02</td>
<td>29.44</td>
<td>6.033E+03</td>
<td>26.73</td>
<td>908.34</td>
</tr>
</tbody>
</table>

It is worthwhile to note that the orbital parameters as the eccentricity and period in Table 1, are
not so accurate but for the purpose of primary knowledge. In fact, the initial orbital elements of
the planets relative to the sun are derived from the planetary ephemerides. To be more specific, the
state vector of the planet relative to the sun at the epoch \( t_0 \) can be extracted from the loaded DE405 ephemeris data, denoted by \((r_{p0}, v_{p0})\) for the sake of presentation. Then, the orbital elements to the sun are obtained by a two-body transformation relationship centred at the sun, for instance, the semi-major axis \( a_{p0} \), eccentricity \( e_{p0} \), inclination \( i_{p0} \), right ascension of the ascending node \( \Omega_{p0} \), argument of periapsis \( \omega_{p0} \), and true anomaly \( f_{p0} \). Obviously, the influence of the planet to the sun is neglected, which is a logical approximation to the real-world problem. Then, the eccentricity is directly passed to the ER3BP model but set to zero in the CR3BP model. After this step, these orbital elements are stored in memory together with the corresponding epoch. The state of the planet forward or backward with respect to the sun is calculated from these parameters based on Keplerian theory. The gravitational parameters of the perturbed bodies are reported in Table 2.\(^{24}\)

<table>
<thead>
<tr>
<th>Perturbation</th>
<th>Earth</th>
<th>Uranus</th>
<th>Neptune</th>
<th>Pluto</th>
<th>Moon</th>
<th>Sun</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_{\text{per}} )</td>
<td>3.986E+05</td>
<td>5.795E+06</td>
<td>6.837E+06</td>
<td>9.816E+02</td>
<td>4.903E+03</td>
<td>1.327E+11</td>
</tr>
</tbody>
</table>

In order to accelerate the computation, this equations in Eq. (1) and Eq. (5) are normalized with the parameters in Table 1. The radius of the target planet is selected as the distance unit, and the time normalization is conducted such that the period of the circular orbit at a distance of the planet’s radius is equal to \( 2\pi \).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Remark</th>
<th>Unit</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( MU )</td>
<td>Gravitational parameter unit</td>
<td>( \text{km}^3/\text{s}^2 )</td>
<td>( \mu_p ) in Table 1</td>
</tr>
<tr>
<td>( LU )</td>
<td>Length unit</td>
<td>( \text{km} )</td>
<td>Radius in Table 1</td>
</tr>
<tr>
<td>( TU )</td>
<td>Time unit</td>
<td>( \text{s} )</td>
<td>( \sqrt{(LU^3/MU)} )</td>
</tr>
<tr>
<td>( VU )</td>
<td>Velocity unit</td>
<td>( \text{km/s} )</td>
<td>( LU/TU )</td>
</tr>
</tbody>
</table>

**Performance Metrics**

The purpose of this paper is to promote a systemic and comprehensive analyses with different dynamical models, as well as the spatial characteristic in comparison to the planar problem. It is imaginable that there are abundant candidate ballistic transfers for options. Therefore, several indices are advanced to assess or categorize the obtained trajectories under a certain background.

- **Capture Ratio \( R_c \):** a massive trajectories have been computed but only a minor part of them can be ballistically captured by the planet. The success of the ballistic capture transfer depends on not only the mutual position of the sun and the planet, but also position of the spacecraft. A success criterion termed Capture Ratio is used to denote the success percentage corresponding to a special condition. By comparing this index, it is expected to know when the spacecraft is likely to be ballistically captured by a planet.
- Stability Index $S$: this parameter is introduced to judge the stability of the orbit relative to a specified body. Note that in two-body problem the higher the Kepler energy is, the less maneuver needs to escape. That is to say, the Kepler energy represents the stability of the orbit. The Jacobi constant can also reflect the stability in the CR3BP model. However, it is inconvenient to seek for an analogous variable that can depict the long-term behavior of the spacecraft in the ER3BP or ephemeris models. A new parameter is introduce to compare the stability of given trajectories, as defined by

$$S = \frac{\Delta T}{n}, \quad n \in \mathbb{Z}^+$$

where $n$ is a counter that records the revolution number around the planet, and $\Delta T$ is the flight time corresponding to the $n$-revolutions. The higher the value of $S$ is, the weaker stability the orbit has. (The definition of a complete revolution will be explained in the next section.)

- Kepler Energy $H = v^2/2 - 1/r$: although the Kepler energy is not able to forecast the future flight in three-body problem, it still contains the combined information relative to the planet. In particular, the terminal Kepler energy shows the gap to the expectant mission orbit.

- Inclination w.r.t. BME@Epoch frame $i$: the inclination in the BME@Epoch frame is important in ensuring the scientific development of space mapping and surveying mission.

- Entry Gate: it is important to obtain the exact entry direction, for instance, from the close proximity to $L_1$ (interior transfer) or $L_2$ (exterior transfer), which is visible in a rotating frame.

**DEFINITION OF BALLISTIC CAPTURE**

In practice, the ballistic capture transfers are sought from massive integrations. Several initial parameters are isometrically discretized and form a high-degree grid, namely, initial condition (IC). The state corresponding to the IC is integrated backward and forward until an appointed ending criterion. The ballistic capture trajectories are obtained by an intersection between the backward forward sets.

**Definition of Initial Condition**

Practical (in terms of the major perturbation from the sun), the definition of the initial condition in the planet-central RTN@Epoch frame is of great expedient for the classification of the stable sets and capture sets, as the reference frame $x_r, y_r, z_r$ in Figure 2(a). Classic orbital elements are used to describe the state at the initial epoch $t_0$, namely, $(r_{p0}, e_0, i_0, \Omega_0, \omega_0, f_0)$, where $r_{p0}$ is the distance of the periapsis, $a_0 = \frac{r_{p0}}{1 - e_0}$. The initial eccentricity $e_0$ is set to 0.95. Analogous to the definitions in Reference 9, 10, 12, the particle is assumed to locate at the periapsis of the osculating ellipse with respect to the planet $(f_0 = 0)$, as reported in Table 4. The epoch $t_0$ corresponds to the periapsis of the planet relative to the sun. Besides the eccentricity and true anomaly, the remaining four elements are scattered in their intervals, such as $r_{p0} \in [0, \text{SOI}]$, $\omega_0 \in [0, 2\pi)$, $i_0 \in [0, \pi)$, and $\Omega_0 \in [0, 2\pi)$. The periapsis distance is linearized into $N_{r_{p0}}$ points, and the value chosen to generate the grid of the other parameters are shown in Table 4, as $\Delta \omega_0$, $\Delta i_0$, and $\Delta \Omega_0$. The last two columns are the forward and backward revolution number, which will be explained in Subsection Construction of Capture Set.
Table 4. Parameter Setting for the Grid of Initial Condition

<table>
<thead>
<tr>
<th>Planet</th>
<th>( t_0 ) (UTC, day)</th>
<th>( t_0 ) (d, m, y)</th>
<th>( e_0 )</th>
<th>( N_{r,0} )</th>
<th>( \Delta \omega_0 ) (°)</th>
<th>( \Delta i_0 ) (°)</th>
<th>( \Delta \Omega_0 ) (°)</th>
<th>( n )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>2458891.70</td>
<td>12 Feb 2020</td>
<td>0.95</td>
<td>548</td>
<td>1</td>
<td>45</td>
<td>45</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Venus</td>
<td>2458928.58</td>
<td>20 Mar 2020</td>
<td>0.95</td>
<td>610</td>
<td>1</td>
<td>45</td>
<td>45</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Mars</td>
<td>2459064.90</td>
<td>03 Aug 2020</td>
<td>0.95</td>
<td>574</td>
<td>1</td>
<td>45</td>
<td>45</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Jupiter</td>
<td>2459965.00</td>
<td>20 Jan 2023</td>
<td>0.95</td>
<td>482</td>
<td>1</td>
<td>45</td>
<td>45</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Saturn</td>
<td>2463565.15</td>
<td>28 Nov 2032</td>
<td>0.95</td>
<td>548</td>
<td>1</td>
<td>45</td>
<td>45</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

![Figure 3. Definition of spatial stability in the perifocal frame.](image)

**Definition of Spatial Stability**

As mentioned in Section Performance Metrics, the spatial stability relies on the definition of the revolution. A pre-specified semi-plane at the initial epoch \( t_0 \) is introduced to define a complete loop (see Figure 3), substituting the line used in the planar problem.9,10 A perifocal frame is defined: x-axis lies in the periapsis direction, z-axis points to the angular momentum vector, and y-axis finishes the right-hand coordinate frame, as \( \vec{x} \vec{y} \vec{z} \) in Figure 3(a)). This semi-plane locates in the \( \vec{x} - \vec{z} \) plane with positive \( \vec{x} \).

Based on this semi-plane, the spacecraft finished a whole revolution once the condition in the following equation is achieved:

\[
\mathbf{r}_t \cdot (\mathbf{h}_0 \times \mathbf{r}_0) = 0 \quad \text{and} \quad \mathbf{r}_t \cdot \mathbf{r}_0 > 0 \quad \text{and} \quad (\mathbf{v}_t \cdot \mathbf{v}_0) \cdot (\mathbf{v}_{t-1} \cdot \mathbf{v}_0) > 0
\]  

(7)

where \( \mathbf{r}_t \) and \( \mathbf{v}_t \) are the current position and velocity vectors, \( \mathbf{v}_{t-1} \) is the velocity vector at previous intersection with the semi-plane, and \( \mathbf{r}_0, \mathbf{h}_0 \) are the position and angular momentum vectors at the initial epoch. It is worth mentioning that the third condition in Eq. (7) is to prevent the repeating interactions with the semi-plane in a short period, termed as incomplete revolutions. The trajectories forward or backward integrated from the initial condition are categorized into four types, as

1. **Unstable (X):** the orbits escape from the planet: \( H_t \geq 0 \) and \( r_t \geq \text{SOI} \).
2. **Crash (K):** the orbits impact the surface of the planet: \( r_t \leq \text{Radius} \).
3. Stable ($W$): the orbits perform a full loop around the planet: Eq. (7).

4. Acrobatic ($D$): the integration do not terminate until the maximum flight time achieved. An interval with four period of the circular orbit in SOI is is adopted in this paper.

**Construction of Capture Set**

The spacecraft that approaches the planet from the infinity point (or far enough in practice) and perform one or more revolutions around it is said to be ballistically captured by the planet.\textsuperscript{12} Therefore, numerous initial condition obtained from previous subsection are integrated forward and backward, and these trajectories that accomplish $n$-revolutions in the forward propagations and escape during the first revolution when integrated backward are stored.\textsuperscript{11, 12} The union of the former initial conditions are called $n$-stable set ($n = 6$ in Table 4), and the latter is $-1$-unstable set ($m = 1$ in Table 4). The capture set is obtained by a manipulation between the two sets, as

$$C_{-1}^n = W_n \cap X_{-1}$$ (8)

**PLANAR ANALYSIS**

The ballistic capture is analyzed in the planar case ($i_0 = 0$ rad) with three different dynamical models: CR3BP, ER3BP, and ephemeris models, according to the parameter settings in Table 4. Without loss of generality, the initial epoch is such selected that the planet locates at the periapsis around the sun, as the second and third columns reported in Table 4. Figure 4 shows the 1-stable, crash, and capture sets in the previous models, and one example trajectory with minimum capture stability index $S_{\text{min}}$ for each model is also illustrated in the RTN@$t_0$ and BRF@$t_0$. The capture sets of the remaining four planets are plotted in Figure 5. Note that the initial conditions corresponds to the prograde osculating ellipses in the RTN@$t_0$ frame.
Figure 4. A comparison of the stable and capture sets in different dynamical models around Mercury (Planar prograde case).
Precisely, the parameters with capture ratio and minimum capture stability index are listed in Table 5. The example trajectories with min $S$ in the ephemeris model are provided for comparison. The results depict that the ER3BP model provides an accurate approximation to the full ephemeris model, but with higher computation speed. Besides, the ballistic capture transfer for the Venus is less unstable than the other four cases.

| Table 5. Capture parameters and orbits in three models (Planar prograde case) |
|---|---|---|---|---|---|
| Planet | Model | Mercury | Venus | Mars | Jupiter |
| $R_c$ (‰) | CR | 0.020 | 0.118 | 0.203 | 0.311 | 0.182 |
| | ER | 0.907 | 0.128 | 0.295 | 0.207 | 0.223 |
| | EP | 0.907 | 0.128 | 0.315 | 0.225 | 0.264 |
| $S_{min}$ (TU) | CR | 3.037E+03 | 7.367E+03 | 1.518E+04 | 5.681E+04 | 7.782E+04 |
| | ER | 1.248E+03 | 7.321E+03 | 1.014E+04 | 3.842E+04 | 7.107E+04 |
| | EP | 1.248E+03 | 7.229E+03 | 1.014E+04 | 3.875E+04 | 7.099E+04 |

The role of the true anomaly plays in the ballistic capture is further analyzed, as shown in Figure 6, Figure 7, Figure 8, and Figure 9. Initial prograde and retrograde situations are compared. In fact, the true anomaly will not affect the ballistic capture transfers in the CR3BP model. These data in the four plots are obtained by the ER3BP model. For the prograde case, the minimum stability index stands in the true anomaly of $\pi/4$, instead of the planet’s periapsis or apoapsis with respect to the sun. However, the situation is more complex for the retrograde case. The minimum $S$ corresponds to the true anomalies of $\pi/4$ and 0 for the Mercury and Venus, respectively. The other three planets is the value of $\pi/2$. Moreover, the distributions of the capture ratio are different in the prograde and retrograde cases, as shown in Figure 8(a) and Figure 9(a).

**SPATIAL ANALYSIS**

Analogous to the planar analysis section, the influence of the initial inclination, and RAAN is explained, as shown in Figure 10. The minimum and maximum value of the capture ratio and stability index are labeled. The trajectories corresponding to the minimum capture stability index in the spatial situation are sorted out. The initial state, $S$, the approach gate (from $L_1$ or $L_2$), and the parameter history are listed in Table 6 and Figure 11. Comparing to the traditional view in two-dimensional space, the extension to the spatial case provides a novel prospect to the ballistic capture problem.

**CONCLUSION**

The role of the dynamical models plays in the ballistic capture transfers is investigated in this paper, including the circular restricted three-body problem, elliptic restricted three-body problem, and the real ephemeris models. Comparing to the traditional planar ballistic capture transfers, an
Figure 5. A comparison of capture sets for Venus, Mars, Jupiter, and Saturn (Planar prograde case).
Figure 6. The role of the true anomaly plays in the capture sets of Mercury (Planar prograde case).

Figure 7. The role of the true anomaly plays in the stable, crash, and capture sets of Mercury (Planar retrograde case).
Figure 8. Capture ratio and capture stability index (Planar prograde case).

Figure 9. Capture ratio and capture stability index (Planar retrograde case).
Figure 10. Capture ratio and capture stability index in spatial case.
Table 6. The parameters of the trajectories with minimum capture stability index

<table>
<thead>
<tr>
<th>Planet</th>
<th>$t_0$ (MJD2000)</th>
<th>Position (LU)</th>
<th>Velocity (VU)</th>
<th>$t_+$ (MJD2000)</th>
<th>$t_-$ (MJD2000)</th>
<th>$S$ (day)</th>
<th>Gate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>7.3472E+03</td>
<td>1.3821 0.6782 0.3296</td>
<td>-0.4055 0.9842 -0.3247</td>
<td>7.4150E+03</td>
<td>7.2981E+03</td>
<td>11.3</td>
<td>$L_1$</td>
</tr>
<tr>
<td>Venus</td>
<td>7.3841E+03</td>
<td>-0.3288 -3.6838 0.9136</td>
<td>0.5845 -0.1477 -0.3852</td>
<td>7.6375E+03</td>
<td>7.1870E+03</td>
<td>42.2</td>
<td>$L_1$</td>
</tr>
<tr>
<td>Mars</td>
<td>7.5204E+03</td>
<td>3.2765 -4.2211 -2.0245</td>
<td>0.4784 0.3094 0.1290</td>
<td>8.1934E+03</td>
<td>7.1535E+03</td>
<td>112.2</td>
<td>$L_1$</td>
</tr>
<tr>
<td>Jupiter</td>
<td>8.4205E+03</td>
<td>12.9378 -3.6962 -1.8992</td>
<td>0.1155 0.3326 0.1397</td>
<td>1.2951E+04</td>
<td>6.3379E+03</td>
<td>755.2</td>
<td>$L_1$</td>
</tr>
<tr>
<td>Saturn</td>
<td>1.2021E+04</td>
<td>13.5986 17.6922 -3.2375</td>
<td>0.0921 -0.0198 0.2786</td>
<td>2.3244E+04</td>
<td>5.3403E+03</td>
<td>1870.6</td>
<td>$L_2$</td>
</tr>
</tbody>
</table>
Figure 11. Trajectories with minimum capture stability index, from the top to bottom rows correspond to the Mercury, ..., and Saturn.
three-dimensional criterion with a predefined semi-plane in the inertial frame is introduced to define the spatial stability. The initial conditions are classified into four types: stable, unstable, crash, and acrobatic. Several metrics, such as the capture ratio, stability index, Kepler energy, and inclination, are addressed to assess the capture capability and performance of each planet. Based on the definition of the spatial stability, the ballistic capture transfers are analyzed in two conditions: planar and spatial cases. The influence of the dynamical models and the true anomaly is presented in the planar analyses. The results show that the ER3BP model is accurate to approximate to the real-world environment for the sun-planet ballistic capture problem. Meanwhile, the best capture transfers correspond to the true anomalies of $\pi/4$ (prograde) and $0 - \pi$ (retrograde), instead of fixing at the periapsis or apoapsis. The spatial results provide a new option to answer the question about “how to capture the spacecraft temporarily or permanently with ballistic transfer trajectory more effective”.

REFERENCES


