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AN IMPROVED INITIAL CONSTRAINT AMONG DIFFERENTIAL ORBITAL ELEMENTS FOR THE J2 INVARIANT RELATIVE MOTION

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A set of analytic constraints among differential orbital elements for J2 invariant relative motion has been obtained in this paper. These new constraints are deduced by nullifying the rates of mean osculating longitude of ascending node difference and the weighted rates' combination of the mean argument difference of perigee and the mean anomaly difference. Here, the related weight is found by checking the newly developed analytic bounds for satellite relative motion. By carrying on a set of simulations, it has found that the modified J2-invariance conditions can lead to better results for the bounded relative motion than the method presented in the historical reference.

I. INTRODUCTION

In practical application, any single satellite cannot run in the ideal Keplerian orbit since it will be affected by some disturbances, like the J2 perturbation. Consequently, the relative motion between two perturbed satellites due to the J2 effects is also not perfectly periodical and bounded. To mitigate the J2 effects for the relative motion, especially in the satellites formation application, many scholars tried to find the possible sets of the initial differential orbital elements. Using the mean elements and the Hamilton dynamics, Schaub et al. [1] firstly found a set of constraints for the differential semi-major axis, the differential eccentricity, and the differential inclination. Then, Gurfil [2] extended the constraints for J2 invariant relative motion by using non-osculating (non-zero gauge) orbital elements. In essence, Schaub et al. obtained the J2 invariant relative motion by nullifying the rates of the osculating mean longitude of ascending node difference and osculating mean argument of latitude difference. However, Gurfil obtained the results by nullifying the rates of non-osculating longitude of ascending node difference and non-osculating mean anomaly. As far, these two solutions are the best results for J2 invariant relative motion in the form of analytic expressions. However, after simulation test, it found that there are still some drifts in the relative motion by using these solutions. Therefore, finding a better solution for J2 invariant relative motion will be a deserved research problem.

The analytic equations for relative positions by the differential orbital elements have been obtained in Reference 3. Using the method in Reference 4, it can simplify the relative positions'

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equations in Reference 3 by transforming the original trigonometric functions with the variable f (i.e. true anomaly) to the algebraic equations with the variable s (representing the tangent function of half the true anomaly). Then, by checking the derivatives of the position components with respect to the variable s, the bounds of the relative motion under the given initial differential orbital elements are determined. Furthermore, by analyzing the expressions of the position components, it has found that the weighted combination of the mean argument difference of perigee and the mean anomaly difference contributes to the bounds of the along-track motion. Here the weights are determined by the eccentricity of the reference satellite. Therefore, by nullifying the rates of osculating longitude of ascending node difference and the weighted rates' combination of the mean argument difference of perigee and the mean anomaly difference, a set of new J2-invariance conditions are obtained.

II. SATELLITE RELATIVE MOTION EQUIATIONS

A. Equations for Satellite Relative Motion

For convenience, a new but essentially little modified satellite relative motion equation will be introduced. Firstly, recall one available solution of the satellite relative dynamics describing by the leader's position and the differential orbital elements of the follower relative to the leader. Here, the leader is defined as the reference satellite, and the follower is another tracking satellite. From Reference 3, one can see that

$$\begin{cases} x = \delta r \\ y = r\left(\delta\theta + \cos i\delta\Omega\right) \\ z = r\left(\sin\theta\delta i - \cos\theta\sin i\delta\Omega\right) \end{cases} \tag{1}$$

where x, y, and z are the position vector's coordinates of the follower relative to the leader. Note here that these coordinates are all described in the local vertical local horizontal (LVLH) frame of the leader. r is the magnitude of the position vector of the leader. θ, i and Ω are the argument of latitude, orbital inclination, and longitude of ascending node, of the leader, respectively. One can easily remember that $\theta = f + \omega$ in which ω is the argument of perigee. A quantity with a prefix δ represents its variation.

Using the basic result on the satellite Keplerian motion,

$$r = \frac{a\left(1-e^2\right)}{1+e\cos f} \tag{2}$$

One can easily obtain the following position variation

$$\delta r = \frac{\left(1-e^2\right)}{1+e\cos f}\delta a - a\frac{2e+\left(1+e^2\right)\cos f}{\left(1+e\cos f\right)^2}\delta e + a\frac{\left(1-e^2\right)}{\left(1+e\cos f\right)^2}e\sin f\delta f \tag{3}$$

To cancel the eccentric anomaly appearing in Eq.(3), one needs to consider the relation between the eccentric anomaly and the true anomaly, which consequently results in the following equation:

$$\delta f = \frac{1}{1-e^2}\left[\frac{1}{\sqrt{1-e^2}}\left(1+e\cos f\right)^2\delta M + \left(2+e\cos f\right)\sin f\delta e\right] \tag{4}$$

Substituting Eq.(3) into Eq.(1) and considering the identity in Eq.(4), the variant solution of satellite relative motion can be written as

$$\begin{cases} x=\dfrac{1-e^2}{1+e\cos f}\delta a-a\cos f\,\delta e+\dfrac{ae\sin f}{\sqrt{1-e^2}}\delta M \\ y=\dfrac{a\left(1-e^2\right)}{1+e\cos f}\left(\delta\omega+\cos i\,\delta\Omega\right)+\dfrac{a\left(2+e\cos f\right)\sin f}{1+e\cos f}\delta e+\dfrac{a\left(1+e\cos f\right)}{\sqrt{1-e^2}}\delta M \\ z=\dfrac{a\left(1-e^2\right)}{1+e\cos f}\sin\left(f+\omega\right)\delta i-\dfrac{a\left(1-e^2\right)}{1+e\cos f}\cos\left(f+\omega\right)\sin i\,\delta\Omega \end{cases} \tag{5}$$

B. An Equivalent Algebraic Form

For convenience, the above solution of satellite relative motion, which is in the trigonometric form of the true anomaly, will be transformed to parametric and algebraic form. Using the same notation in References 4,5, and 6, a new variable s is introduced here, which is the function of the true anomaly,

$$s=\tan\left(f/2\right),\quad f\in\left[0,2\pi\right) \tag{6}$$

It is clear that

$$\sin f=\frac{2s}{1+s^2} \tag{7}$$

$$\cos f=\frac{1-s^2}{1+s^2} \tag{8}$$

So, by using the equalities of Eqs.(7~8), the original solution of satellite relative motion, say Eq.(5), will become

$$\begin{cases} x=c_1+c_6-\dfrac{2ec_6}{\left(1-e\right)s^2+1+e}+\dfrac{2\left(ec_2s-c_1\right)}{s^2+1} \\ y=\left(1-e\right)c_2+c_3+\dfrac{2\left(c_1s+ec_2\right)}{s^2+1}+\dfrac{2\left(c_1s-ec_3\right)}{\left(1-e\right)s^2+1+e} \\ z=-c_5+\dfrac{2\left(c_4s+c_5\right)}{\left(1-e\right)s^2+1+e} \end{cases} \tag{9}$$

where

$$\begin{cases} c_1=a\delta e \\ c_2=a\delta M\big/\sqrt{1-e^2} \\ c_3=a\left(1+e\right)\left(\delta\omega+\delta\Omega\cos i\right) \\ c_4=a\left(1-e^2\right)\left(\delta i\cos\omega+\delta\Omega\sin\omega\sin i\right) \\ c_5=a\left(1+e\right)\left(\delta i\sin\omega-\delta\Omega\cos\omega\sin i\right) \\ c_6=\left(1+e\right)\delta a \end{cases} \tag{10}$$

It should be noted that the coefficients $c_1 \sim c_5$ are exactly same with the ones in References 4 and 6. However, c_6 is a new coefficient since δa is not zero here.

III. BOUNDS THEORY OF SATELLITE RELATIVE MOTION

To obtain the conditions for J2 invariant relative orbits, it is necessary to introduce the bounds theory of satellite relative motion as a preliminary.

A. Bounds of Satellite Relative Motion without Perturbation

It is naturally that the inertial motion of any satellite in the near Earth orbit without perturbation is bounded. This kind of motion is usually called Keplerian motion. Then, consequently, the resulted relative motion of two satellites both in Keplerian orbits will be bounded too. That is to say, there must be a specific limit for the relative motion. In Reference 6, the conclusion has been obtained that the satellite relative motion must be limited between a lower bound and an upper bound in each directional motion, when no perturbation is acted on. The analytic bounds' expressions of the relative motion have been also obtained. In fact, to do that, one only needs to calculate the partial derivative of each coordinate with respect to the variable s. Once the extreme value points are found, the corresponding maximum and minimum will be found by examining the sign of the second order's partial derivatives. Here, the maximum and minimum are the so-called upper and lower bounds, respectively.

Different with Reference 6 which tackles the so-called periodical relative motion, that is to say, $\delta a = 0$, this paper will solve a more general problem. As for the new scenario, i.e., $\delta a \neq 0$, it is clear that the only change locates in the expression of radial directional relative motion. Hence, it is only need to find the new bounds' expression for radial directional relative motion. Considering the small eccentricity, the radial directional relative motion can be rewritten as

$$x \doteq c_1 + c_6 - \frac{\frac{2ec_6}{1-e}}{s^2+1} + \frac{2\left(ec_2 s - c_1\right)}{s^2+1} \tag{11}$$

The partial derivative of x with respect to s is

$$\frac{\partial x}{\partial s} = -2\frac{ec_2 s^2 - 2\left(c_1 + \frac{e}{1-e}c_6\right)s - ec_2}{\left(s^2+1\right)^2} \tag{12}$$

Let $\frac{\partial x}{\partial s} = 0$, then two extreme value points will be found

$$\begin{cases} s_1 = \frac{(1-e)c_1 + ec_6}{e(1-e)c_2} + \frac{1}{1-e}\sqrt{\left[(1-e)c_1 + ec_6\right]^2 + \left[e(1-e)c_2\right]^2} \\ s_2 = \frac{(1-e)c_1 + ec_6}{e(1-e)c_2} - \frac{1}{1-e}\sqrt{\left[(1-e)c_1 + ec_6\right]^2 + \left[e(1-e)c_2\right]^2} \end{cases} \tag{13}$$

The corresponding extreme values of x can be easily determined

$$\begin{cases} x_1 = c_1 + c_6 + \frac{1}{2}\left(\operatorname{sgn}(a_1)\sqrt{a_1^2 + b_1^2} + b_1\right) \\ x_2 = c_1 + c_6 - \frac{1}{2}\left(\operatorname{sgn}(a_1)\sqrt{a_1^2 + b_1^2} - b_1\right) \end{cases} \tag{14}$$

where $\operatorname{sgn}(x)$ is the usual sign function

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases} \tag{15}$$

and

$$a_1 = 2ec_2 \tag{16}$$

$$b_1 = -2c_1 - \frac{2e}{1-e}c_6 \tag{17}$$

By checking the second order's partial derivatives of the above two extreme values, one can easily confirm the following bounds for the radial relative motion

$$\begin{cases} x_{\max} = c_1 + c_6 + \frac{1}{2}\left(\sqrt{a_1^2 + b_1^2} + b_1\right) \\ x_{\min} = c_1 + c_6 - \frac{1}{2}\left(\sqrt{a_1^2 + b_1^2} - b_1\right) \end{cases} \tag{18}$$

Substituting the values of c_1, c_2 and c_6 into Eq.(18), the final expression of the bounds in x direction can be written as

$$\begin{cases} x_{\max} = a\left\{\delta e + (1+e)\frac{\delta a}{a} + \frac{1}{1-e}\left[\begin{array}{l}\sqrt{e^2\frac{1-e}{1+e}(\delta M)^2 + \left[(1-e)\delta e + e(1+e)\frac{\delta a}{a}\right]^2} \\ -\left[(1-e)\delta e + e(1+e)\frac{\delta a}{a}\right]\end{array}\right]\right\} \\ x_{\min} = a\left\{\delta e + (1+e)\frac{\delta a}{a} - \frac{1}{1-e}\left[\begin{array}{l}\sqrt{e^2\frac{1-e}{1+e}(\delta M)^2 + \left[(1-e)\delta e + e(1+e)\frac{\delta a}{a}\right]^2} \\ +\left[(1-e)\delta e + e(1+e)\frac{\delta a}{a}\right]\end{array}\right]\right\} \end{cases} \tag{19}$$

Eq.(19) is the general bounds' expression for the radial relative motion. When $\delta a = 0$, Eq.(19) will degenerate into the one presented in Reference 6.

In a same manner, the bounds of the along-track and cross-track relative motions can be also determined

$$
\begin{cases}
y_{\max} = a\left\{\begin{aligned}&\sqrt{\frac{1-e}{1+e}}\delta M + (1+e)(\delta\omega + \delta\Omega\cos i)\\ &+\frac{1}{1-e}\left\{\begin{aligned}&\sqrt{(2-e)^2\,\delta e^2 + e^2\left[\sqrt{\frac{1-e}{1+e}}\delta M - (1+e)(\delta\omega + \delta\Omega\cos i)\right]^2}\\ &+e\left[\sqrt{\frac{1-e}{1+e}}\delta M - (1+e)(\delta\omega + \delta\Omega\cos i)\right]\end{aligned}\right\}\end{aligned}\right\}\\
y_{\min} = a\left\{\begin{aligned}&\sqrt{\frac{1-e}{1+e}}\delta M + (1+e)(\delta\omega + \delta\Omega\cos i)\\ &-\frac{1}{1-e}\left\{\begin{aligned}&\sqrt{(2-e)^2\,\delta e^2 + e^2\left[\sqrt{\frac{1-e}{1+e}}\delta M - (1+e)(\delta\omega + \delta\Omega\cos i)\right]^2}\\ &-e\left[\sqrt{\frac{1-e}{1+e}}\delta M - (1+e)(\delta\omega + \delta\Omega\cos i)\right]\end{aligned}\right\}\end{aligned}\right\}
\end{cases}
\tag{20}
$$

$$
\begin{cases}
z_{\max} = a(1+e)\left\{-(\delta i\sin\omega - \delta\Omega\cos\omega\sin i) + \frac{1}{1+e}\left[\sqrt{\delta i^2 + \delta\Omega^2\sin^2 i} + (\delta i\sin\omega - \delta\Omega\cos\omega\sin i)\right]\right\}\\
z_{\min} = a(1+e)\left\{-(\delta i\sin\omega - \delta\Omega\cos\omega\sin i) - \frac{1}{1+e}\left[\sqrt{\delta i^2 + \delta\Omega^2\sin^2 i} - (\delta i\sin\omega - \delta\Omega\cos\omega\sin i)\right]\right\}
\end{cases}
\tag{21}
$$

where Eqs.(20~21) are exactly same with the ones in Reference 6.

B. Effects of Non-zero Differential Semi-major Axis

It is clear that, when $\delta a \neq 0$, the differential mean anomaly will linearly increase with the time *t*.

$$
\delta M = \delta M_0 + \delta\dot{M}\cdot t = \delta M_0 + \sqrt{\frac{\mu}{a^3}}\frac{\delta a}{a}t \tag{22}
$$

If the perturbations existed in Earth orbit are ignored, the leader's orbital elements and the other five differential orbital elements of the follower with respect to the leader will be constant. In this case, the fact of $\delta a \neq 0$ will lead to the increase of the bounds in radial and along-track relative motions. Therefore, for the relative motion without perturbation, bounded and consequently periodical relative motion will strongly demand that $\delta a = 0$.

In fact, the real environment shows many disturbances and perturbations, which will severely affect the bounds of the relative motion. Therefore, in the following discussions, the important perturbation, J2 effects of the Earth, will be included. The study also shows that, when J2 perturbation is considered, $\delta a = 0$ is not necessary to obtain a bounded relative motion.

C. Effects of J2 Perturbation

Under the J2 perturbation, many terms related to the relative motion will be no longer constant. In fact, from Reference 7, the following equations are easily determined.

$$\begin{cases} \dot{\bar{a}} = 0 \\ \dot{\bar{e}} = 0 \\ \dot{\bar{i}} = 0 \\ \dot{\bar{\Omega}} = -\gamma a^{-\frac{7}{2}} \left(1-e^2\right)^{-2} \cos i \\ \dot{\bar{\omega}} = \gamma a^{-\frac{7}{2}} \left(1-e^2\right)^{-2} \left(2 - \frac{5}{2}\sin^2 i\right) \\ \dot{\bar{M}} = \mu^{\frac{1}{2}} a^{-\frac{3}{2}} + \gamma a^{-\frac{7}{2}} \left(1-e^2\right)^{-\frac{3}{2}} \left(1 - \frac{3}{2}\sin^2 i\right) \end{cases} \tag{23}$$

where the term with a bar on its head denotes its mean value in a whole period under the J2 perturbation. Correspondingly, this set of elements is called mean elements. Here, the term γ is defined as follows

$$\gamma = \frac{3}{2} J_2 a_E^2 \mu^{\frac{1}{2}} \tag{24}$$

where μ is the Earth gravity constant, a_E the equatorial radius, and J_2 the coefficient of J2 perturbation.

The variations of the mean elements' derivatives can be represented as

$$\begin{cases} \delta\dot{\bar{a}} = 0 \\ \delta\dot{\bar{e}} = 0 \\ \delta\dot{\bar{i}} = 0 \\ \delta\dot{\bar{\Omega}} = \frac{\partial\dot{\bar{\Omega}}}{\partial a}\delta a + \frac{\partial\dot{\bar{\Omega}}}{\partial e}\delta e + \frac{\partial\dot{\bar{\Omega}}}{\partial i}\delta i \\ \delta\dot{\bar{\omega}} = \frac{\partial\dot{\bar{\omega}}}{\partial a}\delta a + \frac{\partial\dot{\bar{\omega}}}{\partial e}\delta e + \frac{\partial\dot{\bar{\omega}}}{\partial i}\delta i \\ \delta\dot{\bar{M}} = \frac{\partial\dot{\bar{M}}}{\partial a}\delta a + \frac{\partial\dot{\bar{M}}}{\partial e}\delta e + \frac{\partial\dot{\bar{M}}}{\partial i}\delta i \end{cases} \tag{25}$$

where the partial derivatives of $\frac{\partial y}{\partial x}$ ($y = \dot{\bar{\Omega}}, \dot{\bar{\omega}}, \dot{\bar{M}}$, $x = a, e, i$) can be found in Appendix. The results in Appendix indicate that these partial derivatives are all constant. This means that the resulted rates of the mean elements' variations are constant too.

From Eq.(25), one can conclude that, under the J2 perturbation, only three mean elements' variations, namely $\delta\Omega$, $\delta\omega$ and δM, will vary with time. And one important finding is that the rates of these three terms are constant, which will support the following discussion for J2 invariant relative orbits.

IV. IMPROVED INITIAL CONDITIONS FOR J2 INVARIANT RELATIVE ORBITS

A. Strict Conditions

According to the discussion above, one can easily find a way to resist the J2 effects for the relative orbits by nullifying the rates of the variations of $\delta\Omega$, $\delta\omega$ and δM. Notes that these three rates are the linear functions of the other three variations, namely, δa , δe and δi . Hence, a natural concept is to let the following identities hold

$$\delta\dot{\Omega} = 0, \quad \delta\dot{M} = 0, \quad \delta\dot{\varpi} = 0 \tag{26}$$

or, equivalently,

$$\delta a = 0, \quad \delta e = 0, \quad \delta i = 0 \tag{27}$$

This solution can be also found in References 1 and 2, as a general discussion of J2 invariant relative orbits. The above condition for generating the J2 invariant relative orbits is also called *three constraints* in Reference 7. It is clear that this condition is a very strict way to keep the bounded relative motion under the J2 perturbation in long-term period. Nevertheless, it is no doubt that the strict condition is one of the best ways to form the J2 invariant relative motion.

B. Loose Conditions

Considering the limits of the above strict conditions for generating possible formation configurations, many researchers proposed other loose conditions. The main idea is still to nullify the rates of the elements' variations between two satellites as much as possible. One way called *two constraints* in Reference 7, which is first presented in Reference 1, is to nullify the rates of the right ascension and the mean argument of latitude.

$$\delta\dot{\Omega} = 0, \quad \delta\dot{M} + \delta\dot{\varpi} = 0 \tag{28}$$

The same treatment is also appeared in Reference 2, but in a different dynamics frame and using the gauge velocity concept. This proposition is relatively intuitive. It can be easily confirmed that the varying term in cross-track relative motion is only $\delta\Omega$. Therefore, nullifying the rate of $\delta\Omega$ is quite justifiable. However, there is a question for the other equality in Eq.(28). Because in mathematical expressions related to the relative motion, there is no place to show that these two terms, i.e., δM and $\delta\omega$ are always appeared in a directly additive way. In the next section, a possible improvement will be presented.

C. Modified Loose Conditions

When the analytic bounds for the relative motion are obtained, one can easily determine that if the bounds under J2 perturbation are constant, the related relative motion will be also J2 invariant. Using this concept, the conditions for J2 invariant relative orbits can be easily found. Notes that the changing terms that will affect the bounds are only three, i.e. $\delta\Omega$, $\delta\omega$, δM . In these three terms, the mean anomaly difference δM only occur in *x* and *y* directions. However, the effect of δM in *x* direction is far less than the one in *y* direction due to the weight of eccentricity. Therefore, the change caused by δM in *x* direction can be ignored. Thus, to find the conditions for J2 invariant relative orbits, the following constraints will be necessary.

$$\dot{y}_{\max} = 0, \quad \dot{z}_{\max} = 0 \tag{29}$$

where the derivatives of $y_{\max}$ and $z_{\max}$ are

$$\dot{y}_{\max} = a\left(\xi_{11}\delta\dot{M} + \xi_{12}\delta\dot{\omega} + \xi_{12}\cos i\delta\dot{\Omega}\right) \tag{30}$$

$$\dot{z}_{\max} = a\xi_{13}\delta\dot{\Omega} \tag{31}$$

where

$$\xi_{11} = \frac{1}{\sqrt{1-e^2}} \left\{ 1 + \frac{e^2 \left[\sqrt{\frac{1-e}{1+e}} \delta M - (1+e)(\delta\omega + \delta\Omega \cos i) \right]}{\sqrt{(2-e)^2 \delta e^2 + e^2 \left[\sqrt{\frac{1-e}{1+e}} \delta M - (1+e)(\delta\omega + \delta\Omega \cos i) \right]^2}} \right\} \tag{32}$$

$$\xi_{12} = \frac{1+e}{1-e} \left\{ 1 - 2e - \frac{e^2 \left[\sqrt{\frac{1-e}{1+e}} \delta M - (1+e)(\delta\omega + \delta\Omega \cos i) \right]}{\sqrt{(2-e)^2 \delta e^2 + e^2 \left[\sqrt{\frac{1-e}{1+e}} \delta M - (1+e)(\delta\omega + \delta\Omega \cos i) \right]^2}} \right\} \tag{33}$$

$$\xi_{13} = \left\{ e \cos\omega \sin i + \frac{\delta\Omega \sin^2 i}{\sqrt{\delta i^2 + \delta\Omega^2 \sin^2 i}} \right\} \tag{34}$$

Substituting Eqs.(30~31) into Eq.(29) leads to

$$\delta a = -\frac{J_2}{2L^4\eta^5} \frac{4+3\beta\eta}{\beta} \left(1 + 5\cos^2 i\right) \frac{ae}{\eta} \delta e \tag{35}$$

$$\delta i = \frac{4e}{\left(1-e^2\right)\tan i} \delta e \tag{36}$$

where

$$\beta = \sqrt{\frac{1-e}{(1+e)^3}} \left\{ \frac{1 + \frac{e^2 \left[\sqrt{\frac{1-e}{1+e}} \delta M - (1+e)(\delta\omega + \delta\Omega \cos i) \right]}{\sqrt{(2-e)^2 \delta e^2 + e^2 \left[\sqrt{\frac{1-e}{1+e}} \delta M - (1+e)(\delta\omega + \delta\Omega \cos i) \right]^2}}}{1 - 2e - \frac{e^2 \left[\sqrt{\frac{1-e}{1+e}} \delta M - (1+e)(\delta\omega + \delta\Omega \cos i) \right]}{\sqrt{(2-e)^2 \delta e^2 + e^2 \left[\sqrt{\frac{1-e}{1+e}} \delta M - (1+e)(\delta\omega + \delta\Omega \cos i) \right]^2}}} \right\} \tag{37}$$

$$L = \sqrt{a/a_E}, \quad \eta = \sqrt{1-e^2} \tag{38}$$

Eqs.(35~36) are the modified conditions for J2 invariant relative orbits. When comparing these new results with the ones in Reference 1, it can be seen the only difference is the weight of β .

V. SIMULATION RESULTS AND ANALYSIS

A. Simulation Conditions

To illustrate the newly developed conditions for J2-invariance, a set of configuration space simulations of the invariant motion will be completed in this section. To this end, let the initial mean orbital elements of the leader be

$$a = 7153\text{km}\ ,\ i = 0.838\text{rad}\ ,\ \Omega = 0\ ,\ \omega = 0.52\text{rad}\ ,\ M_0 = 0 \tag{39}$$

where the eccentricity in the following simulation cases will be shown subsequently.

The initial differential mean orbital elements are as follows:

$$\delta e = 0.01\ ,\ \delta\Omega = 0.05\text{rad}\ ,\ \delta\omega = 0.01\text{rad}\ ,\ \delta M_0 = -0.02\text{rad} \tag{40}$$

where δa and δi will be calculated using Eqs.(35~36).

To compare the results of this paper with the ones in Reference 1, the J2 invariant conditions of there are also shown below

$$\delta a = -\frac{J_2}{2L^4\eta^5}(4+3\eta)\left(1+5\cos^2 i\right)\frac{ae}{\eta}\delta e \tag{41}$$

$$\delta i = \frac{4e}{\left(1-e^2\right)\tan i}\delta e \tag{42}$$

which will be used to calculate δa and δi for another simulations.

B. Results and Discussion

Using the methods of Reference 1 and the modified method developed in this paper, the relative motion of two satellites are propagated in 50 orbital periods. The results are presented below. Figures 1 and 2 illustrate the histories of the relative motion by using the J2-invariance conditions developed in this paper. It can be seen that the resulted relative motion is bounded and quasi-periodic. This demonstrates that the new method is valid. Table 1 and Figure 3 presented the coordinates and inter-satellite bounds' changes by using the method in Reference 1 and the current method, respectively. One can see that the bounds' changes using the modified method are almost always less than the ones using the method in Reference 1. This means the current modified method has better performance for achieving the J2-invariant relative orbits.

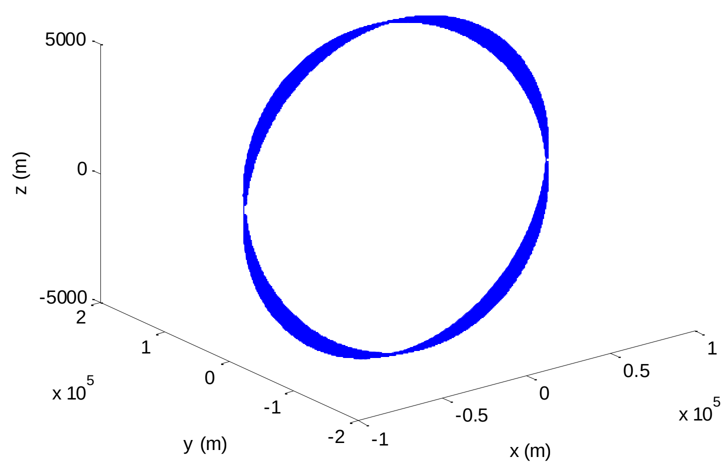

Figure 1. Histories of the relative motion by using the J2-invariance conditions developed in this paper (e=0.01).

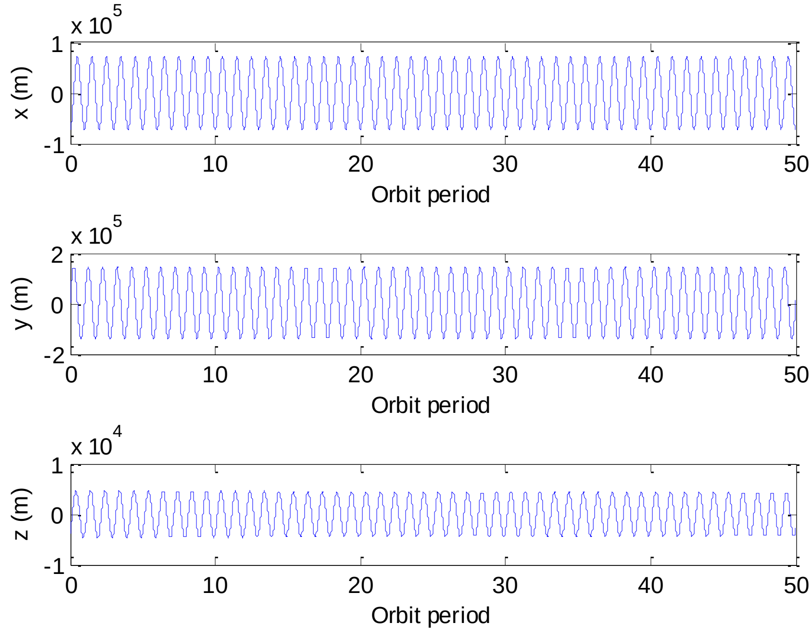

Figure 2. Histories of the relative motion by using the J2-invariance conditions developed in this paper (e=0.01).

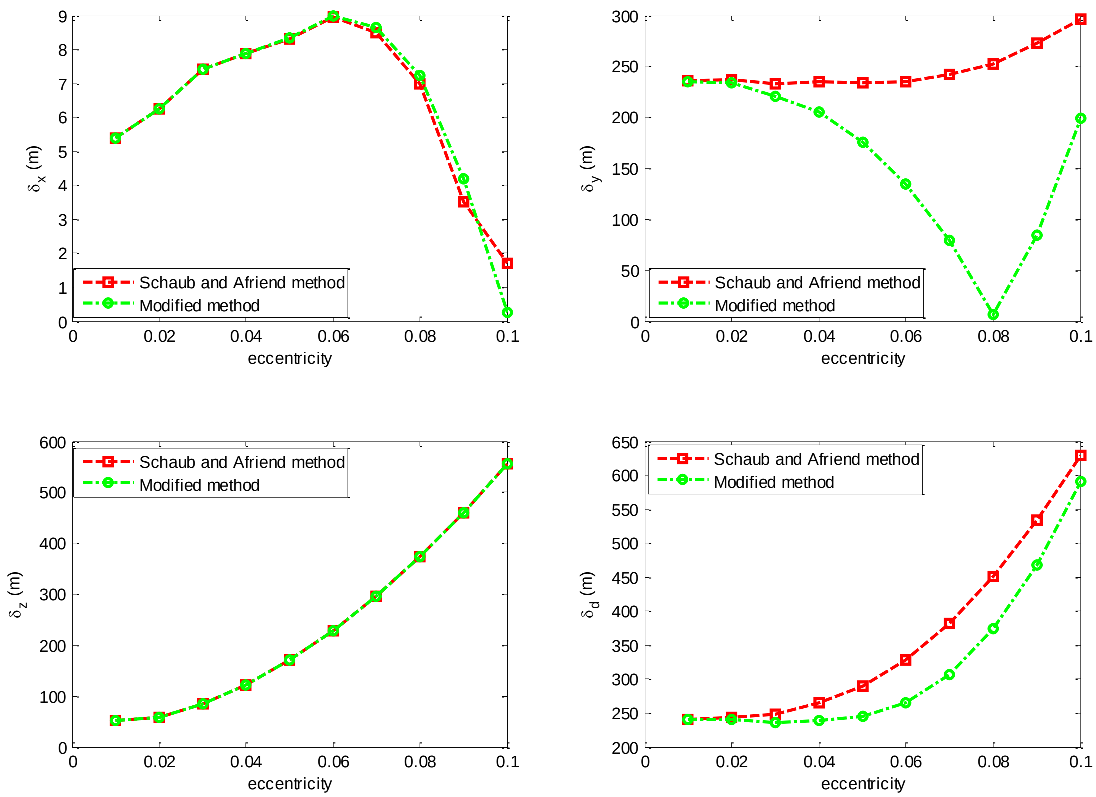

Figure 3. Changes of the bounds when applying the J2-invariance conditions.

Table 1. Changes of the bounds when applying the J2-invariance conditions.

Eccentricity	Schaub and Alfriend's method [1]				Modified method			
	δx	δy	δz	δd	δx	δy	δz	δd
0.01	6.258478	237.2753	57.90896	244.3198	6.257859	233.6894	57.90872	240.8388
0.02	7.429478	233.2284	83.98139	247.9991	7.42946	221.0079	83.98068	236.5427
0.03	7.86653	234.4964	122.1451	264.5182	7.869239	205.2386	122.1432	238.964
0.04	8.322043	233.6764	170.5418	289.4105	8.332001	175.8923	170.5377	245.1339
0.05	8.961458	235.4261	228.7588	328.3845	8.986867	134.3826	228.7511	265.4553
0.06	8.496469	241.8853	296.452	382.7068	8.637835	79.52353	296.4391	307.0419
0.07	6.982549	252.8872	373.4355	451.0596	7.239742	6.926958	373.4155	373.5499
0.08	3.515967	272.4068	459.7435	534.3986	4.192924	84.2491	459.7144	467.3893
0.09	1.69794	296.272	555.4322	629.5116	0.247557	199.4321	555.3915	590.1127
0.10	5.37877	235.6142	51.41348	241.2185	5.378704	235.1699	51.41345	240.7845

VI. CONCLUSIONS

Using the bounds theory, the J2-invariance conditions developed in this paper are more illustrative and convincing than the ones in the references. The simulation results also confessedly prove that the modified J2-invariance conditions have better performance for the bounded relative motion than the Schaub and Alfriend's conditions. However, there is still a limit in this current research. Since the bounds theory is valid and much more effective for the small eccentricity case, the modified J2-invariance conditions inherit this character and perform well in the small eccentricity case too. To extend the current results to the large eccentricity case, it is necessary to improve the bounds theory firstly. Nevertheless, the current work has a potential chance to be applied into the cluster flying mission. Therefore, for the small elliptical reference satellite formation, the conditions are well justified to be used as the bounded flying conditions. This will save much fuel consumption for cluster flying mission.

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APPENDIX: ORBITAL ELEMENTS' VARIATION UNDER J2 PERTURBATION

The mean elements' variations under J2 perturbation are shown below.

$$\frac{\partial \dot{\Omega}}{\partial a} = \frac{7}{2}\gamma a^{-\frac{9}{2}}\left(1-e^2\right)^{-2}\cos i \tag{43}$$

$$\frac{\partial \dot{\Omega}}{\partial e} = -4e\gamma a^{-\frac{7}{2}}\left(1-e^2\right)^{-3}\cos i \tag{44}$$

$$\frac{\partial \dot{\bar{\Omega}}}{\partial i} = \gamma a^{-\frac{7}{2}} \left(1-e^2\right)^{-2} \sin i \tag{45}$$

$$\frac{\partial \dot{\bar{\omega}}}{\partial a} = -\frac{7}{2}\gamma a^{-\frac{9}{2}} \left(1-e^2\right)^{-2} \left(2-\frac{5}{2}\sin^2 i\right) \tag{46}$$

$$\frac{\partial \dot{\bar{\omega}}}{\partial e} = 4e\gamma a^{-\frac{7}{2}} \left(1-e^2\right)^{-3} \left(2-\frac{5}{2}\sin^2 i\right) \tag{47}$$

$$\frac{\partial \dot{\bar{\omega}}}{\partial i} = -5\gamma a^{-\frac{7}{2}} \left(1-e^2\right)^{-2} \sin i \cos i \tag{48}$$

$$\frac{\partial \dot{\bar{M}}}{\partial a} = -\frac{7}{2}\gamma a^{-\frac{9}{2}} \left(1-e^2\right)^{-\frac{3}{2}} \left(1-\frac{3}{2}\sin^2 i\right) - \frac{3}{2}\sqrt{\mu} a^{-\frac{5}{2}} \tag{49}$$

$$\frac{\partial \dot{\bar{M}}}{\partial e} = 3e\gamma a^{-\frac{7}{2}} \left(1-e^2\right)^{-\frac{5}{2}} \left(1-\frac{3}{2}\sin^2 i\right) \tag{50}$$

$$\frac{\partial \dot{\bar{M}}}{\partial i} = -3\gamma a^{-\frac{7}{2}} \left(1-e^2\right)^{-\frac{3}{2}} \sin i \cos i \tag{51}$$

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