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A POLAR VARIABLES VIEW ON THE CRITICAL INCLINATION PROBLEM

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The frozen-perigee behavior of orbits at the critical inclination is usually displayed after an averaging procedure. However, this singularity in Artificial Satellite Theory manifests also in the presence of short-period effects. Indeed, a closed form expression relating orbital inclination and the ratio anomalistic draconitic frequencies is derived for the main problem, which demonstrates that the critical inclination results from commensurability between the fequencies with which the radial and polar variables evolve in the instantaneous plane of motion. This relation also shows that the critical inclination value is slightly modified by the degree of oblateness of the attracting body, as well as by the orbit's size and shape.

INTRODUCTION

The critical inclination problem of Artificial Satellite Theory (AST), a particular orbital inclination that "freezes" the perigee of elliptic orbits, has been qualified as "the most celebrated problem in AST".[1] This special inclination, of 63.4 degrees (resp. 116.6 deg.), makes apparent from the simple derivation of Lagrange planetary equations when the disturbing function of the main problem of AST, which is obtained after neglecting from the geopotential all harmonic coefficients except for the second order zonal harmonic (J_2), is averaged over the mean anomaly (see Sec. 10.6 of Ref. 2, for instance). However, this simple averaging misses important effects of the second order of J_2. Indeed, for a given semi-major axis, second order effects reduce the number of elliptic frozen orbits at the critical inclination to just four isolated solutions with arguments of the perigee 0, $\pi/2$, π, and $3\pi/2$, respectively, two of which are stable and the other two unstable[3,4] —the phase portrait may change depending on the number of zonal harmonics included in the geopotential[5,6] or when lunisolar perturbations are included in the model.[7] Aerospace engineers very soon found practical applications for satellite missions with this inclination, as is the case of the celebrated Molniya orbits (see, Ref. 8, for instance). More recently the critical inclination has been identified as a suitable choice for deploying cluster missions requiring bounded satellite motion.[9]

From a mathematical perspective, the critical inclination problem is commonly presented as a singularity in the solution of the motion of a massless particle in the potential of an oblate planet. This singularity is caused by the appearance of small divisors in the

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analytic integration of the secular terms of a perturbation theory developed by averaging (see Sec. 12 of Chap. 17 of Ref. 10, for instance). However, since practitioners had not found the expected increase in the coordinates perturbations of orbits close to the critical inclination,[11] on the one hand, and because the problem of small divisors is related with the occurrence of resonances between different frequencies of the motion, on the other, which is fairly clear in the case of planetary or tesseral resonances[12,13] but was not so apparent in the analytic integration of the main problem, some controversy arised on wether the singularity produced by the critical inclination was just virtual or not, which was extended for some time.[1,14] But the singularity produced by the critical inclination is, undoubtedly, essential. Indeed, following the lines of global geometric solutions proposed by Deprit,[15,16] it was demonstrated by Cushman[17,18] and Coffey et al.,[19] with later amendments by Ferrer et al.,[20] that the critical inclination phenomenon is produced by a change in the stability of circular orbits in a bifurcation event, a result in agreement with the behavior that had been anticipated by Izsak.[21]

Remark that the troubles in integrating orbits at, or close to, the critical inclination only emerge in the analytical approach, whereas there are no difficulties in the numerical integration of orbits at the critical inclination. Furthermore, the small denominators problem is avoided in practical implementations of Brouwer's-type analytical integration by using special ways for handling the critical terms (see Section 7 of Ref. 22 or appendix A.F. of Ref. 23). Even these days, analogous methods are proposed to cope with virtual singularities, as the cases of small eccentricities or inclinations, and the critical inclination.[24]

Most mentioned research efforts base on averaging procedures, a fact that may prompt the belief that the critical inclination singularity only discloses in the treatment of the secular terms of the gravity potential. However, the bifurcation event is checked to make apparent too from the direct numerical integration of the main problem, including both short- and long-period effects.[25] Furthermore, the occurrence of critical inclinations in the analytical integration of spherical variables intermediaries of the main problem was also well-known.[26–28] For the latter, the critical inclination was clearly identified with a 1 to 1 commensurability between the satellite's draconitic (from node to node) and anomalistic (from perigee to perigee) periods, thus providing a clear physical explanation of the resonance phenomenon. Namely, at resonance the sub-satellite point always reaches a given latitude in the same time interval, and each time this happens the satellite's altitude over the surface of the Earth is exactly the same. Therefore, because of the axial symmetry of the potential, the satellite suffers an identical gravitational pull after a constant period.

Classical intermediary orbits can be integrated in closed form, but at the expense of using elliptic integrals. Therefore, critical inclinations are only uncovered after a series expansion of corresponding closed form solutions. On the contrary, Deprit's radial intermediary[29] accepts a closed form solution which is free from elliptic integrals. This solution is used here to show that orbit inclination can be written explicitly as a function of the frequencies of the motion without need of resorting to averaging or series expansions. In this way, it is possible to demonstrate that resonances between polar variables in the instantaneous orbital plane are parametrized by inclination. Besides, for the small values of J_2 characteristic of

AST, it is shown that the 1 to 1 resonance is the unique resonance that can be guaranteed to exist. Finally, the existence of other critical inclinations is illustrated, which may exist for much higher values of J_2.

THE RADIAL INTERMEDIARY

Artificial satellite theory studies the motion of a massless particle in the presence of the gravitational potential. For Earth-like bodies, the second order zonal harmonic coefficient (J_2) dominates all other harmonic coefficients, and hence the truncation of the expansion of the gravitational potential by neglecting all harmonic coefficients except J_2 is customarily called the *main problem* of AST.

The main problem Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r}\left[1 - J_2\,\frac{\alpha^2}{r^2}\left(\frac{1}{2} - \frac{3}{4}\sin^2 i + \frac{3}{4}\sin^2 i\cos 2\theta\right)\right], \tag{1}$$

where the gravitational parameter μ, the scaling factor α, and the oblateness coefficient J_2 are physical parameters that define the gravity field, i is orbit inclination, $\cos i = N/\Theta$, and the set of polar-nodal variables $(r, \theta, \nu, R, \Theta, N)$ stand for radius from the earth's center of mass, argument of latitude, argument of the node, radial velocity, modulus of the angular momentum vector, and polar component of the angular momentum vector, respectively.

The fact that ν is cyclic in Eq. (1), and consequently N is an integral, reflects the axial symmetry accepted by the main problem, which decouples the reduced flow in the instantaneous plane of motion

$$\frac{\mathrm{d}(r,\theta)}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial(R,\Theta)}, \qquad \frac{\mathrm{d}(R,\Theta)}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial(r,\theta)}, \tag{2}$$

of two degrees of freedom, from the rotation of the node

$$\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial N}, \tag{3}$$

which may be integrated by quadrature after solving he differential system in polar variables given in Eq. (2).

In spite of the reduced problem remains non-integrable, a variety of integrable approximations of the main problem, the so-called *intermediaries*, have been proposed in the literature. In particular, this note deals exclusively with Deprit's natural, radial intermediary

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{Q^2}{r^2}\right) - \frac{\mu}{r}, \tag{4}$$

where $Q \equiv Q(\Theta, N)$ is a constant function given by

$$Q = \Theta\sqrt{1 + \sigma\left(\frac{1}{2} - \frac{3}{2}\cos^2 i\right)}, \tag{5}$$

and σ is given by

$$\sigma = J_2 \frac{\alpha^2}{p^2}, \tag{6}$$

with $p = \Theta^2/\mu$ standing for the orbit *semilatus rectum*. Equation (4) is obtained by applying the elimination of the parallax transformation to Eq. (1) up to the first order of J_2, and converts the main problem into a quasi-Keplerian, integrable problem.[29] For the reader's convenience, the contact transformation that converts the main problem Hamiltonian into Deprit's radial intermediary Eq. (4) is provided in the appendix. Note that we provide different, simpler expressions than those given in Deprit's original paper. Indeed, in view of the recent demonstration that the elimination of the parallax transformation is much more affordable in the setting provided by Delaunay variables, the use of the C and S parallactic functions does not provide any value to this canonical transformation. Besides, the expressions provided in the appendix are better tuned for a fast evaluation which may be an essential prerequisite in practical applications (cf. 30, this meeting).

A brief outline of the conventional integration of the Hamiltonian flow of Eq. (4) is given in Ref. 29, whereas the alternative integration by means of a *torsion* is also provided with much more detail. The latter being a transformation in implicit variables, Deprit reconstructed the torsion as an explicit transformation by means of a Lie transform which, up to the first order of J_2, shows that the torsion in the orbital plane becomes the identity mapping at the critical inclination. Here we take a different approach, in which, without need of resorting to series expansions, the trajectory solution in polar variables is used to demonstrate that orbit inclination can be expressed as a function of the ratio between the draconitic and anomalistic periods. This closed form relation is then used to compute the (critical) inclination at which both periods get the same value, therefore meeting the frozen orbit condition.

Trajectory in the instantaneous plane of motion

Because the argument of latitude is cyclic in Deprit's radial intermediary in Eq. (4) the modulus of the angular momentum is constant, and, therefore, the orbital plane evolves with constant inclination in the transformed phase space. Then, the one degree of freedom problem in (r, R) decouples from the motion of θ and ν. The reduced problem is easily integrated by noting that, from Hamilton equations,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial R} = R. \tag{7}$$

Then, for any manifold $\mathcal{H} = h$, replacing Eq. (7) into Eq. (4), leads to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{2h + 2\frac{\mu}{r} - \frac{Q^2}{r^2}}. \tag{8}$$

Equation (8) is in separate variables, and the independent variable is easily integrated by quadrature. Besides,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \Theta} = \frac{P}{r^2}, \tag{9}$$

where $P \equiv P(\Theta, N)$ is a constant function given by

$$P = \Theta \left[1 - \sigma \left(\frac{1}{2} - 3\cos^2 i\right)\right]. \tag{10}$$

Therefore, the equation of the trajectory in the instantaneous plane of motion is obtained from Eqs. (9) and (8) as

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{P}{r^2 \sqrt{2h + 2(\mu/r) - (Q^2/r^2)}}, \tag{11}$$

which is also in separate variables and hence can be integrated by quadrature to give

$$\theta = P \int_{r_m}^{r} \frac{\mathrm{d}s}{s^2 \sqrt{2h + 2(\mu/s) - (Q^2/s^2)}}, \tag{12}$$

where r_m is the minimum value of the radial distance.

Equation (12) is solved by the standard change of variable from the radius to the true anomaly v. Indeed, making

$$r = \frac{Q^2/\mu}{1 + e\cos v}, \tag{13}$$

where

$$e^2 = 1 + (2h/\mu)\,(Q^2/\mu), \qquad 0 \le e \le 1, \tag{14}$$

and taking into account that $r = r_m \Rightarrow v = 0$, one trivially gets

$$\theta = \theta_0 + \frac{P}{Q}\,v. \tag{15}$$

Hence, from Eq. (13),

$$r = \frac{Q^2/\mu}{1 + e\cos[(Q/P)(\theta - \theta_0)]}. \tag{16}$$

Critical inclinations

By differentiation of Eq. (15)

$$\frac{n_r}{n_\theta} = \frac{Q}{P} = k(\Theta, N). \tag{17}$$

where $n_\theta = \mathrm{d}\theta/\mathrm{d}t$ is the draconitic frequency, that is $n_\theta = 2\pi/T_\theta$ where T_θ is the time elapsed between two consecutive satellite passages by the ascending node, and $n_r = \mathrm{d}v/\mathrm{d}t$ is the anomalistic frequency, that is $n_r = 2\pi/T_r$ where T_r is the time elapsed between two consecutive perigee passages.

In general, P and Q will be incommensurable numbers, and the trajectory on the instantaneous orbital plane, Eq. (16), will be a rosette. However, according Eq. (17), for rational values of k the frequencies n_θ and n_r will become commensurable, and, therefore,

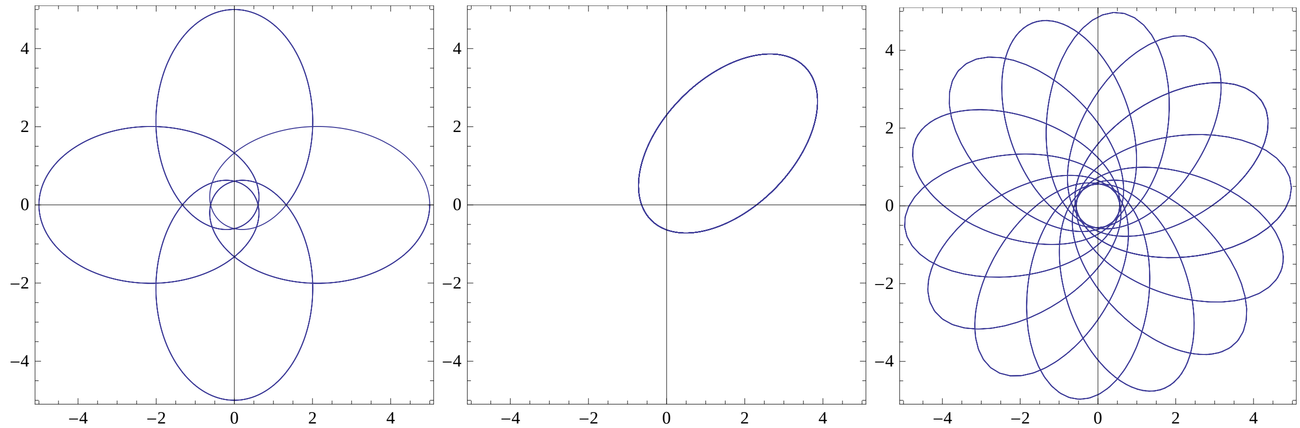

Figure 1. Sample solutions of Eq. (16) for $Q/P = 4/5$ (left), 1 (center), and $14/13$ (right).

the rosette will be a periodic orbit which closes after a number of periods, as illustrated in Fig. 1 for, from left to right, $Q/P = 4/5$, 1, and $14/13$ ($e = 0.8$, $\mu = Q^2$, $\theta_0 = 3\pi/4$).

Now, dividing Eq. (5) by Eq. (10), and using Eq. (17)

$$\frac{n_r}{n_\theta} = \frac{\sqrt{1 + \sigma\left(\frac{1}{2} - \frac{3}{2}\cos^2 i\right)}}{1 - \sigma\left(\frac{1}{2} - 3\cos^2 i\right)}, \tag{18}$$

from which orbit inclination can be written explicitly as a function of the frequencies of the reduced problem. Namely,

$$\cos^2 i = \frac{\sqrt{1 + 4(6+\sigma)\,(n_r/n_\theta)^2} - 1 - 2(2-\sigma)\,(n_r/n_\theta)^2}{12\sigma\,(n_r/n_\theta)^2}. \tag{19}$$

Leaving aside the case of almost rectilinear orbits, following ressults limit to non-impact orbits $p \geq \alpha$. In these cases, σ is small and arbitrary rational values $(n_r/n_\theta) = k$ will not result, in general, in real inclinations producing periodicity in the instantaneous plane of motion. Therefore, one must explore carefully which rational values of k are allowed in Eq. (19).

Expansion of Eq. (19) in power series of σ gives

$$\cos^2 i = \frac{\sqrt{1+24k^2} - 1 - 4k^2}{12k^2}\,\frac{1}{\sigma} + \frac{1}{6} + \frac{1}{6\sqrt{1+24k^2}} - \frac{k^2}{6(1+24k^2)^{3/2}}\,\sigma + \mathcal{O}(\sigma^2) \tag{20}$$

which clearly shows that $k^2 = 1$, corresponding to the 1 to 1 resonance between θ and r, is the only value of k that cancels the coefficient of σ^{-1} —the first summand of Eq. (20). In this case, Eq. (19) is simplified to

$$\cos^2 i_{\mathrm{c}} = \frac{1}{6} - \frac{5}{12\sigma}\left(1 - \sqrt{1 + \frac{4}{25}\sigma}\right), \tag{21}$$

which shows the dependence of the critical inclination on the J_2 value and also on the angular momentum, cf. Eq. (6). The limit $\sigma \to 0$ results in the well-known value of the critical inclination $\cos^2 i_c = 1/5$. For the Earth, in which case $J_2 = \mathcal{O}(10^{-3})$, it is found that this critical inclination value is accurate to the order of J_2^2. Indeed,

$$\cos^2 i_c = \frac{1}{5} - \frac{1}{750}\sigma + \frac{1}{9375}\sigma^2 + \mathcal{O}(\sigma^3), \tag{22}$$

where $(1/750) \sim J_2 \sim \sigma$.

Besides, because $\theta = \omega + v$, where ω is the argument of the perigee, Eq. (15) trivially shows that the perigee becomes fixed (or "frozen") at $\omega = \theta_0$ for the 1 to 1 resonance. The degeneracy of the solution (all orbits become frozen at the critical inclination) is a result of the integrability of the Hamiltonian truncation to the first order of J_2 in Eq. (4). However, as it is well-known, second order effects of J_2 break this degeneracy to leave only a discrete number of frozen orbits (see Ref. 19, and references therein).

Other possible resonances k rational would also require $k^2 \approx 1$ to compensate the smallness of σ in the first summand of Eq. (20) so that the condition $\cos^2 i \leq 1$ can be fullfiled. Because this only might happen for higher order resonances $k \sim 1 + \mathcal{O}(\sigma)$, corresponding inclinations, if they exist, will not cause any small divisor type complication in a perturbation theory, and hence are not of major concern in AST.

Conversely, assumed that higher values of J_2 may exist, other critical inclinations can be found. Thus, for instance, for $\sigma = 0.1$ ($J_2 = \mathcal{O}(0.1)$) resonances $n_r/n_\theta = 19/25$, $4/5$, $1/1$, and $14/13$, will result in critical inclinations at 3.75, 23.66, 63.44, and 86.34 deg, respectively. This is illustrated in Fig. 2, where the ratio n_r/n_θ, as given in Eq. (18), is shown as a function of σ for different inclinations. Ordinates $n_r/n_\theta = 0.76$, 0.8, 1, and 1.077 correspond to the mentioned resonances for the abscissa $\sigma = 0.1$. Remark that none of the inclination lines in Fig. 2 is a straight line, not even the $63.44\,\text{deg}$ one.

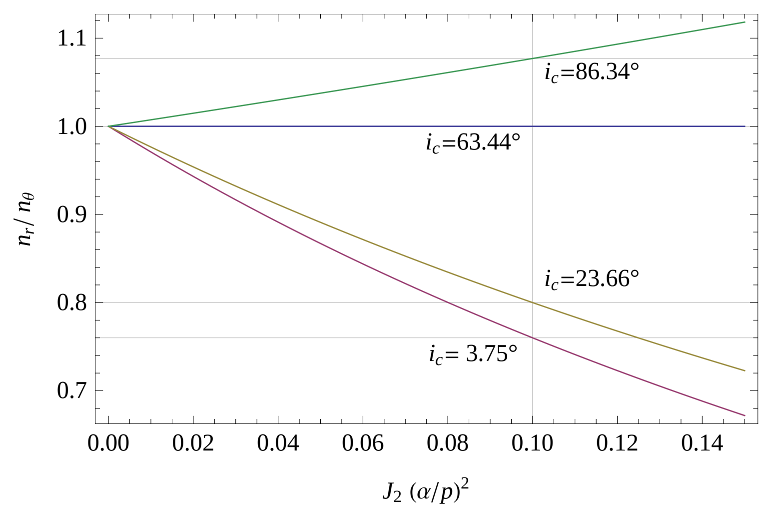

Figure 2. Evolution of $k = (n_r/n_\theta)$ with $\sigma = J_2\,(\alpha/p)^2$ for different inclinations, cf. Eq. (19).

CONCLUSIONS

The mapping between orbit inclination and the ratio draconitic anomalistic frequencies of the main problem of AST may be made explicit in both directions in closed form. This lets look at the small divisors problem arising in the vicinity of the critical inclination as the familiar problem of resonances between the different frequencies of the motion, which in this particular case happen between the rates of variation of both polar motion variables. For those inclinations leading to resonance the orbit in the instantaneous plane of motion turns into a closed rosette. In the particular case of the 1 to 1 resonance the trajectory in the orbital plane becomes a mere ellipse which, therefore, has the perigee frozen in spite of its non-Keplerian character.

Analogously to the case of tesseral resonances, where resonant motion is parametrized by semi-major axis independently of orbit inclination, in axial-symmetric force fields resonant satellite motion is parametrized by inclination, independently of orbit semimajor axis. For the latter, in the case of Earth-like bodies the only relevant case is the 1 to 1 resonance. However, there are no mathematical objections for other critical inclinations of the main problem to happen assumed that the J_2 coefficient may take much higher values than those of Earth-like bodies.

APPENDIX: TRANSFORMATION TO THE RADIAL INTERMEDIARY

Equation 4 is achieved after the elimination of the parallax —a canonical transformation from original polar variables $(r,\theta,\nu,R,\Theta,N)$ to new variables $(r',\theta',\nu',R',\Theta',N')$— is carried out up to the first order of J_2, and, then, it should be written in prime variables. Corresponding transformation equations are

$$\begin{aligned}
\frac{r-r'}{p'} &= \kappa\left(1-\frac{3}{2}s'^2-\frac{1}{2}s'^2\cos 2\theta'\right)\\
\theta-\theta' &= \kappa\left\{\left[\frac{3}{4}-\frac{5}{4}c'^2-(1-3c'^2)\frac{p'}{r'}\right]\sin 2\theta'+\frac{p'R'}{\Theta'}\left[1-6c'^2+(1-2c'^2)\cos 2\theta'\right]\right\}\\
\nu-\nu' &= \kappa\, c'\left[\left(\frac{1}{2}-2\,\frac{p'}{r'}\right)\sin 2\theta'+\frac{p'R'}{\Theta}\,(3+\cos 2\theta')\right]\\
\frac{R-R'}{\Theta'/p'} &= \kappa\,\frac{p'^2}{r'^2}\,s'^2\sin 2\theta'\\
\frac{\Theta-\Theta'}{\Theta'} &= \kappa\, s'^2\left[\left(\frac{1}{2}-2\,\frac{p'}{r'}\right)\cos 2\theta'-\frac{p'R'}{\Theta'}\sin 2\theta'\right]\\
N &= N'
\end{aligned}$$

where $\kappa=-\frac{1}{2}\sigma=-\frac{1}{2}J_2\,\alpha^2/p'^2$, cf. Eq. (6), $p'=\Theta'^2/\mu$, and $s'\equiv\sin i'$, $c'\equiv\cos i'=N/\Theta'$.

Note that, instead of taking the original transformation equations from Ref. 29 (p. 133), we borrowed the transformation equations from Ref. 30 because of their simplicity: they only require the evaluation of sine and cosine functions of argument 2θ. These simpler expressions were rendered after realizing that the so-called parallactic functions C and S, two invariants of the Keplerian motion pertaining to the kernel of the Lie derivative in the algebra of functions in which the original elimination of the parallax is based, are not necessary at all. Indeed, straightforward derivations show the higher efficiency of the elimination of the parallax when approached in Delaunay variables.[31,32] Then, after computing this Hamiltonian simplification[33] in the Delaunay chart, the transformation equations of the elimination of the parallax are easily formulated in the more convenient set of polar-nodal variables by means of standard relations.

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