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ON OPTIMIZATION OF PARAMETERS FOR LINEAR STABILIZATION SYSTEMS

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This work deals with numerical methods of parameter optimization for linear stabilization systems. We formulate a special mathematical programming problem in terms of matrix inequalities. Solving this problem we get optimal parameters for a stabilizer. The developed methodology is illustrated by an example concerning optimization of parameters for a satellite stabilization system.

INTRODUCTION

Several aspects of spacecraft projects require parameter optimization for stabilization systems. A number of approaches to this problem has been suggested. Usually the principal difficulty to overcome is the necessity to combine a fast damping, on one hand, with keeping the system transient trajectory close to its nominal motion on the other hand, as it happens, the above qualities somehow contradict each other.

In previous papers, the authors have suggested a numerical approach that allows one to solve the optimization problem keeping a balance between the above characteristics of the transient process. The numerical algorithm described in[3] yields a flexible and effective solution. However, its implementation requires a large number of estimations of the objective function and therefore is quite time-consuming.

Here we suggest a modification of the above approach based on consecutive estimations of Lyapunov function for the system in question, which allows one to considerably reduce the calculation time of parameter optimization maintaining its efficiency. As an example, we consider parameter optimization of a gravity-gradient stabilizer and compare the efficiency of optimization methods.

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STATEMENT OF THE PROBLEM

Consider linear differential equation

$$\dot{x} = A(u)x,\ x \in R^n, t \geq 0, \tag{1}$$

where $u \in U \subset R^k$ is a parameter. It is assumed that the zero equilibrium position of system (1) is asymptotically stable whenever $u \in U$. Let $\{\lambda_1(u), \cdots, \lambda_n(u)\}$ be the roots of the characteristic equation of system (1). The degree of stability of (1) is defined by

$$\delta(u) = -\max_{i=\overline{1,n}} \operatorname{Re} \lambda_i(u).$$

The parameter u should be chosen to optimize, in some sense, the behaviour of the trajectories. It's impossible to construct a stabilizer that is optimal in all aspects. For example, for a linear controllable system, the pole assignment theorem guarantees the existence of a linear feedback yielding a linear differential equation with any given set of eigenvalues. One can choose a stabilizer with a very high damping speed, i.e. solving the following problem

$$\begin{aligned} \delta(u) &\to \max, \\ u &\in U. \end{aligned} \tag{2}$$

Unfortunately such a stabilizer is practically useless because of so-called peak-effect (see[4,10]). Namely, there exists a large deviation of the solutions from the equilibrium position at the beginning of the stabilization process, whenever the module of the eigenvalues is big. The aim of this work is to develop a numerical tool oriented to the minimization of the maximal deviation with constrained damping speed. A direct approach to these problems was described in.[3] Recently, a new method based on Linear Matrix Inequalities (LMI) techniques[2] was proposed in.[6] The method consists of construction of invariant ellipsoids for the linear system using semidefinite programming (SDP) algorithms. The authors of[6] considered the system with $A(u) = C + bu^T$ and studied the problem of the maximal deviation minimization with constrained damping speed:

$$\begin{aligned} &\alpha \to \min, \\ &PA(u)^T + A(u)P \preceq -2\mu P, \\ &I \preceq P \preceq \alpha I. \end{aligned}$$

(Here $\mu > 0$ is fixed.) In this work we consider a more general case $A(u) = C + \sum_j b^j (u^j)^T$. Moreover the vector parameters u^j are subject to parallelepiped constraints, $u = (u^1, \ldots, u^J) \in U = U^1 \times \ldots \times U^J$. The use of LMI techniques allows us to significantly reduce the CPU time needed to optimize the parameters of stabilizers. We illustrate the method with some examples.

DIRECT APPROACH VERSUS LMI TECHNIQUES

Recall the direct approach to the problem of minimization of maximal deviation.[3] Denote by $x(t, x_0, u)$ the solution to the Cauchy problem

$$\begin{aligned} &\dot{x} = A(u)x,\ x \in R^n,\ t \in [0, T], \\ &x(0) = x_0. \end{aligned} \tag{3}$$

The problem of minimization of the maximal deviation can be formalized as follows:

$$\begin{aligned} &\max_{t \in [0,T]} \max_{|x_0|=1} |x(t, x_0, u)| \to \min, \\ &\max_{|x_0|=1} |x(T, x_0, u)| \le \delta, \\ &u \in U, \end{aligned} \tag{4}$$

i.e., as a problem of determination of parameters that correspond to minimization of the maximum deviation of trajectories and satisfy certain restrictions at the final moment of time.

Let $\varepsilon > 0$ be small enough. We approximate problem (4) by the following problem

$$\begin{aligned} &\bar{\varphi}_0 \longrightarrow \min, \\ &\left|\tilde{x}(t_k^i, x_j^i, u)\right|_i \le \bar{\varphi}_i + \varepsilon,\ i = \overline{0, m}, \\ &u \in U, \end{aligned} \tag{5}$$

where $t_0^i = 0$, $t_k^i \in \Delta_i$, $x_j^i \in B_i$, $j = \overline{1, J}$, and

$$\tilde{x}(t_{k+1}^i, x_j^i, u) = \tilde{x}(t_k^i, x_j^i, u) + \tau A(u)\tilde{x}(t_k^i, x_j^i, u),\ \tau = t_{k+1}^i - t_k^i,\ k = \overline{0, N},$$

is the Euler approximation for the solution $x(\cdot, x_j^i, u)$. Problem (4) can be approximated by problems (5) with any given accuracy.

The most involved part of this approach is the computing of the maximal deviation:

$$D(u) = \max_{t \in [0,T]} \max_{|x_0|=1} |x(t, x_0, u)|\,.$$

This functional can be substituted by its elliptic approximation. Namely, denote by $F(u)$ the value of the following SDP problem:

$$\begin{aligned} &\alpha \to \min, \\ &PA(u)^T + A(u)P \preceq 0, \\ &I \preceq P \preceq \alpha I. \end{aligned}$$

The matrix P defines the Lyapunov function $V(x) = \langle x, P^{-1}x \rangle$ for system (3) (see e.g.,[5]). The asphericity, $\sqrt{F(u)}$, of the ellipsoid $\{x \mid V(x) = 1\}$ can be used as an approximated value of $D(u)$. Let $\sigma_0 > 0$. We consider the problem

$$\begin{aligned} &F(u) \to \min, \\ &\sigma(A(u)) \geq \sigma_0, \\ &u \in U, \end{aligned}$$

and solve it using combination of penalty function and derivative-free methods, e.g. Nelder-Mead method. In what follows we illustrate the possibility of this approach by some examples. Namely, we study the difference between the values of the objective functions $D(u)$ and $\sqrt{F(u)}$, and between optimal parameters found analytically and via minimization of $F(u)$.

EXAMPLES

In this section we consider two examples of parameter optimization for mechanical system.

Oscillator

Solving the problem

$$\alpha \to \min, \tag{6}$$
$$P(A + bK^*)^* + (A + bK^*)P \preceq 0, \tag{7}$$
$$I \preceq P \preceq \alpha I. \tag{8}$$

it is possible to give a trivial solution to the minimal overshooting problem for the oscillator.[9] Set

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad P = \begin{pmatrix} u & w \\ w & v \end{pmatrix}, \quad K = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Following[6] show that there exist vectors K such that I solves Lyapunov inequality (7). This implies that the optimal values are $\hat{\alpha} = 1$ and $\hat{P} = I$. Indeed, with $P = I$ inequality (7) takes the form

$$\begin{pmatrix} 0 & 1+\xi \\ 1+\xi & 2\eta \end{pmatrix} \preceq 0.$$

This condition is equivalent to $\xi = -1$ and $\eta \leq 0$. The corresponding eigenvalues of the closed-loop system are $\lambda = (\eta \pm \sqrt{\eta^2 - 4})/2$, $\eta < 0$. From the engineering point of view, obviously, the best choice is $\eta = -2$. In this case the system is asymptotically stable, has maximum degree of stability, does not oscillate, and has minimal overshooting equal to 1.

Numerical optimization We apply the developed method to choose parameters for the oscillator. The corresponding system has the matrix $A(u) = C + bu^T$, where

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$b = (0,1)^T,$$
$$u = (-u^1, -u^2)^T, \;\; -2.15 \le u^1 \le 0.25, \; -6 \le u^2 \le -1.$$

The results are presented in the following table.

σ_0	u^1	u^2	$\sqrt{F(u)}$	$D(u)$	$\frac{\lvert\sqrt{F(u)}-D(u)\rvert}{D(u)}$
-1.0	1.00000	1.00000	1.0000	1.0000	0
-0.5	1.00000	1.15640	1.0000	1.0000	0

Table 1. Optimization of parameters of oscillator

Note that the method finds the right values of parameters in both cases. The corresponding values of $\sqrt{F(u_{\rm opt})}$ coincide with $D(u)$.

Optimal parameters for satellite-stabilizer system

Dynamics of a system of connected bodies moving in a circular orbit has been subject of several studies.[7,8] Two-body connected system can be considered as a model of spacescraft with gravity-gradient or aerodynamical stabilizer. If the oscillations are relatively small, the motion of the system is described by a linear model.

Consider the motion of a connected two-body system in a circular orbit around the Earth. Body 1 is a satellite with the center of mass O_1, body 2 is a stabilizer with the center of mass O_2. These two bodies are linked to each other at the point P through a dissipative hinge mechanism (Figure 1). Let O be the center of mass of the system.

We use three reference frames: $OXYZ$ is the orbital coordinate frame, its axis OZ is directed along the radius vector of the point O with respect to the center of the Earth, OX is directed along the velocity of the point O, and OY is normal to the orbit plane. The axes of referential frames $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$ are the central principal axes of inertia for bodies 1 and 2 respectively. Consider motion of the system in the orbit plane supposing that the bodies are connected in their centres of mass, i.e., the points O_1, O_2, O, and P coincide. Let α_1 and α_2 be the angles between the axis OX and the axes O_1x_1 and O_2x_2 respectively. Denote by $\alpha'_i, i = 1, 2$, the derivative of α_i with respect to time. The equations of motion for this system can be written as[7]

$$B_1\,\alpha''_1 + 3\,\omega_0^2\,(A_1 - C_1)\sin\alpha_1\,\cos\alpha_1 + \bar{k}_1\,(\alpha'_1 - \alpha'_2) = 0,$$
$$B_2\,\alpha''_2 + 3\,\omega_0^2\,(A_2 - C_2)\sin\alpha_2\,\cos\alpha_2 - \bar{k}_1\,(\alpha'_1 - \alpha'_2) = 0.$$

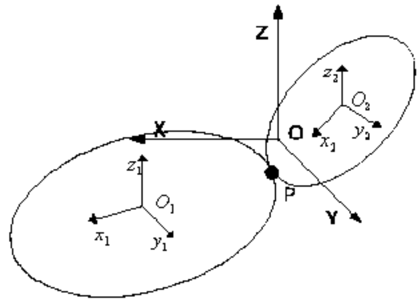

Figure 1. The satellite stabilizer system

Here A_1, B_1, C_1 and A_2, B_2, C_2 are the principal moments of inertia of the bodies, $\bar{k}_1$ is the damping coefficient of the system, and ω_0 is the constant angular velocity of the orbital motion of the system's center of mass. Introducing a new independent variable $\tau = \omega_0 t$ and the dimensionless parameters

$$p_1 = \frac{A_1 - C_1}{B_1},\ p_2 = \frac{A_2 - C_2}{B_2},\ \mu = \frac{B_2}{B_1},\ k_1 = \frac{\bar{k}_1}{\omega_0 B_1},$$

the equations of motion can be written as

$$\begin{aligned} &\ddot{\alpha}_1 + 3p_1 \sin\alpha_1 \cos\alpha_1 + k_1(\dot{\alpha}_1 - \dot{\alpha}_2) = 0, \\ &\ddot{\alpha}_2 + 3p_2 \sin\alpha_2 \cos\alpha_2 - \tfrac{k_1}{\mu}(\dot{\alpha}_1 - \dot{\alpha}_2) = 0. \end{aligned} \tag{9}$$

Here the dot denotes the derivative with respect to τ. The parameters $(p_1,\, p_2,\, k_1,\, \mu)$ satisfy the following conditions

$$-1 \le p_1 \le 1, \quad -1 \le p_2 \le 1, \quad \mu > 0, \quad k_1 > 0. \tag{10}$$

We study small oscillations of system (9) in the vicinity of the equilibrium position

$$\alpha_{10} = 0,\ \alpha_{20} = 0.$$

The equations of motion, linearized in the vicinity of the above stationary solution, take the form

$$\begin{aligned} &\ddot{\alpha}_1 + 3p_1\alpha_1 + k_1(\dot{\alpha}_1 - \dot{\alpha}_2) = 0, \\ &\ddot{\alpha}_2 + 3p_2\alpha_2 - \tfrac{k_1}{\mu}(\dot{\alpha}_1 - \dot{\alpha}_2) = 0. \end{aligned} \tag{11}$$

The characteristic equation for system (11) is

$$\mu\,\lambda^4 + k_1\,(1+\mu)\,\lambda^3 + 3\,\mu\,(p_1 + p_2)\,\lambda^2 + 3\,k_1(\,p_1 + \mu\,p_2)\,\lambda + 9\,\mu\,p_1\,p_2 = 0. \tag{12}$$

Analysis of (12) allows one to obtain the necessary and sufficient conditions of asymptotic stability. The region of asymptotic stability is given by

$$\{(k_1,\, p_1,\, p_2,)\,:\, k_1 > 0,\, p_1 > 0,\, p_2 > 0,\, p_1 \neq p_2\}. \tag{13}$$

Taking into account the feasibility conditions for the system parameters, we arrive at the following set of admissible parameters for our optimization problem:

$$U = \{(p_1,\, p_2,\, k_1,\, \mu) \,:\, k_1 > 0,\, \mu > 0,\, 0 < p_1 \le 1,\, 0 < p_2 \le 1,\, p_1 \ne p_2\}. \tag{14}$$

The maximal degree of stability Consider the set U described by (14). Let $u = (p_1,\, p_2,\, k_1,\, \mu)$ be a parameter belonging to the set U. Let $\{\lambda_1(u), \cdots, \lambda_4(u)\}$ be the roots of equation (12). The inclusion $u \in U$ implies that $\operatorname{Re}\lambda_i(u) < 0$, $i = \overline{1,4}$. In[1] and[8] it is proved that the maximal degree of stability is achieved when all the roots of the characteristic equations are real and equal. This situation becomes possible only when the conditions

$$\begin{aligned} k_1 &= 4\delta(u)\frac{\mu}{1+\mu}, \\ p_1 + p_2 &= 2\delta^2(u), \\ 3(p_1 + \mu p_2) &= (1+\mu)\delta^2(u), \\ 9p_1p_2 &= \delta^4(u) \end{aligned}$$

are satisfied. The above system has two sets of solutions:

$$\begin{aligned} \hat{p}_1 &= (3-2\sqrt{2})^2 \simeq 0.0294, \\ \hat{p}_2 &= 1, \\ \hat{k}_1 &= \sqrt{6}(3-2\sqrt{2}) \simeq 0.4203, \\ \hat{\mu} &= 3-2\sqrt{2} \simeq 0.1716, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \hat{p}_1 &= 1, \\ \hat{p}_2 &= (3-2\sqrt{2})^2 \simeq 0.0294, \\ \hat{k}_1 &= \sqrt{6} \simeq 2.4495, \\ \hat{\mu} &= 3+2\sqrt{2} \simeq 5.8284. \end{aligned} \tag{16}$$

Numerical optimization We apply the developed method to construct stabilizers for the system. The corresponding system has the matrix $A(u) = C + b^1(u^1)^T + b^2(u^2)^T$, where

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$b^1 = (0,0,1,0)^T,$$
$$b^2 = (0,0,0,1)^T,$$
$$u^1 = (-u^{11}, 0, -u^{12}, u^{12})^T, \quad -3 \le u^{11} \le 3, \; u^{12} \ge 0,$$
$$u^2 = (0, -u^{21}, u^{22}, -u^{22})^T, \quad -3 \le u^{21} \le 3, \; u^{22} \ge 0.$$

σ_0	u^{11}	u^{21}	u^{12}	u^{22}	$\sqrt{F(u)}$	$D(u)$	$\frac{\lvert\sqrt{F(u)}-D(u)\rvert}{D(u)}$
-0.1	0.84376	1.20031	0.16146	0.19258	1.0919	1.0622	0.0280
-0.2	0.71546	1.48948	0.30668	0.44250	1.1993	1.1316	0.0598
-0.3	0.60841	1.91851	0.42822	0.73844	1.3270	1.2092	0.0974
-0.4	0.54599	2.27614	0.50614	1.03341	1.4149	1.2588	0.1240
-0.5	0.49951	2.80701	0.57477	1.36253	1.5198	1.3195	0.1518
-0.6	0.15675	2.50209	0.47480	1.89695	1.9917	1.7076	0.1664
-0.7	0.08791	3.00466	0.41974	2.45422	2.4145	1.9186	0.2585

Table 2. Optization of parameters for satellite-stabilizer system

The last line in Table 2 approximately corresponds to case (15). The value of σ_0 is the maximal possible under the imposed constraints. Although the values of $D(u_{\text{opt}})$ and $\sqrt{F(u_{\text{opt}})}$ are rather different, the difference between the corresponding parameters is negligible.

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