FORMATION FLYING DYNAMICS OF MICRO-SATELLITES NEAR EQUATORIAL LOW ORBITS UNDER THE INFLUENCE OF J2 AND J3

Harijono Djojodihardjo[1]

The present work assess the effect of the Earth's oblateness parameters, particularly J_2 and $\mathbf{J_3}$, on the formation flight of micro satellites in near equatorial low orbits. The modified Hill-Clohessy-Wiltshire equations are extended to include the J_2 and $\mathbf{J_3}$ effects in the LVLH frame of reference. The primary gravitational perturbation effect is due to the equatorial bulge term, J_2. The J_2 term changes the orbit period, a drift in perigee, a nodal precession rate and periodic variations in all the orbital elements. The J_2 zonal harmonic captures the equatorial bulge of the Earth, and is the largest coefficient describing the Earth's shape. There is about a $21km$ difference in equatorial and polar radii due mainly to this bulge. In the Earth orbit about 800 km altitude, the J_2 effect is much larger in comparison with other perturbations such as atmospheric drag, solar radiation pressure and electro-magnetic effects. The modified linearized Hill-Clohessy-Wiltshire to take into account the influence of J_2 and J_3 is utilized to determine the orbits of twin spacecraft in formation flight in Near Equatorial orbits, where the variation of J_2 is less apparent. A simplified approach, capitalizing on the balance between linearized approach and expected fidelity of the obtained solution, has been synthesized to arrive at a linearized J_2 and J_3 modified HCW equation. The computational results obtained are assessed by comparison to Schweighart-Sedwick formula. The significance and relevance of the influence of these parameters in the determination and design of formation flying orbits are assessed through parametric study. As a particular example, for low earth orbit (i.e. 847 km), the error is about 0.25km from the desired relative position in the LVLH or Hill frame after 16.67 hours. Further comparison to similar results in the literature exhibits the plausibility of the work.

Key words : ***Formation Flying, Gravitational Potential, J_2 , J_3 , Near-Equatorial Low Orbits, Orbital Mechanics***

INTRODUCTION

Research on autonomous satellite formation flying has been of great interest and need in the last few decades, and multiple spacecraft to replace a single large satellite will be an enabling technology for many missions to come. These clusters of spacecrafts usually work together to accomplish a mission, and are of great interest for interferometry, space-based communications, and missions to study the magnetosphere, to cite a few examples. The benefits to using multiple spacecraft include increased productivity, reduced mission and

[1] Professor, Aerospace Engineering Department, Faculty of Engineering, Universiti Putra Malaysia
harijono@djojodihardjo.com

launch costs, graceful degradation, on-orbit reconfiguration options, and the ability to carry out missions that would not be possible otherwise.

The need to accurately determine and control the position of satellites within the formation have relied on the Hill's equations, also known as the Clohessy-Wiltshire equations, due to their linearity and simplicity, and is referred here as the Hill-Clohessy-Wiltshire's (HCW) equation. Hill-Clohessy-Wiltshire's linearized equations of relative motion are a set of linearized equations that describe the relative motion of two spacecraft in similar near-circular orbits assuming Keplerian central force motion. With the need for better fidelity in the results of the linearized approach, in particular to take into account relevant disturbances in low Earth orbits such as the influence of the Earths's oblateness, atmospheric disturbances and the like, there is a need to modify the Hill- Clohessy-Wiltshire's linearized equations. Many researchers have introduced modifications to capture the effect of the dominant Earths's oblateness parameter on the formation flying satellite cluster.

The Earth gravitational potential can be represented by spheroidal harmonics (References 1, 2, 3 and 4). Linearizing the gravitational terms in the presence of the dominant oblateness parameter, J_2 , for the deputy satellite with respect to the chief's reference orbit, analytical solutions similar to that of the *HCW* equations can be derived. These J_2 *–Linearized Modified Hill's Equations* describe the mean motion changes in both the in-plane and out-of-plane motion, taking into account the linearization assumptions (References 5, 6, 7 and 8). A new set of constant coefficient, linearized differential equations of motion can be obtained, which are similar in form to Hill-Clohessy-Wiltshire's equations, but they capture the effects of the J2 disturbance force. Other researcher (e.g. Reference 9) has also introduced modifications to the HCW equations in similar fashion to include the effects of atmospheric drag while maintaining their linearity and simplicity.

With the growing trend toward reducing the size of satellites, careful modeling of the perturbation forces on formation flying of Micro-Satellites in low orbits is necessary. Then the objective of the present work is the investigation of Earth's oblateness effects on the formation flight of micro satellites in low circular orbits and the assessment of their influence on formation maintenance. In order to study spacecraft formation flight in low Earth orbits, in addition to the effect of J2, the present paper will also investigate the additional influence of the next term J3 of the gravity potential. For this purpose, the Hill-Clohessy-Wiltshire's equations which have been modified by Schweighart and Sedwick (References 5 and 6) as a set of linearized differential equations of motion to include the J2 effects will be extended to incorporate the effects of J3 for circular orbits while maintaining their linearity and simplicity. In addition to low circular orbits, the extension will include reference orbits of small eccentricity. Particular attention is given to near equatorial orbits, where J2 can be considered to be constant. Numerical simulation results based on such model will be presented to reveal the effects of each of the two dominant terms characterizing the *Earth Oblateness Effects* on the formation flight of micro satellites in low near equatorial circular orbits. Parametric studies are carried out with respect to orbital inclination and radius, and chief-deputy satellites separation distance. Results obtained will be discussed with reference to the influence of other disturbing forces that have been presented in the literature.

The influence of the oblateness parameter J2 changes the nature of the orbit of the Chief satellite taken as the reference orbit than that approximated by the Hill-Clohessy-Wiltshire equation, which has been meticulously elaborated by Schweigert and Schweigert and Sedwick. The rederivation of the satellite relative motion incorporate correction of the orbital period of the reference orbit, the *nodal drift* of the reference orbit and the cross-track motion, taking into account the initial conditions. The present work introduced further simplifications by considering near equatorial orbits.[2]

---

[2] Closed form solutions for the influence of oblateness parameter has been worked out by Gurfil (references 13 and 14).

COORDINATE SYSTEM TRANSFORMATION

Referring to Coordinate System conventionally utilized (References 4, 10 and 11), here the subscript N denotes a vector in the ECI frame, and a subscript O denotes a vector in the satellite-centered frame. The (r −θ − i) coordinate system (or Earth Centered Chief Satellite Orbital Plane coordinate system) is used in describing the J_2 disturbance in the local (x - y - z) coordinate system. The two Euler angles, θ and i , and r complete the associated geometrical transformation from the ECI frame to the (r −θ − i) frame, where the direction cosine matrix is formed by the 3-1-3 Euler angle sets Ω , i and θ .

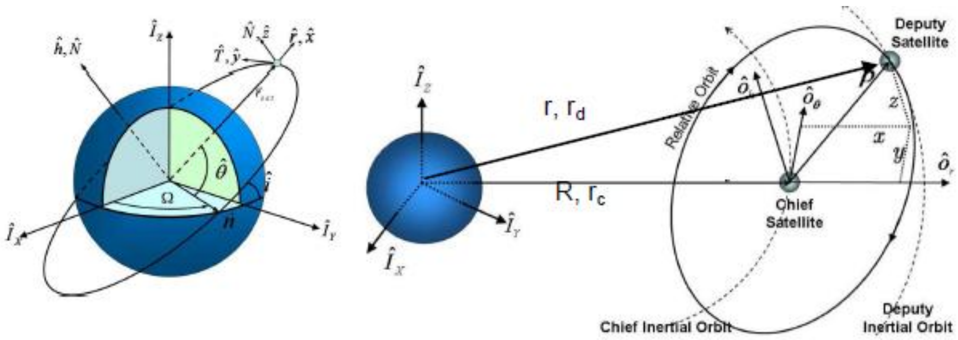

Figure 1. The (r −θ − i) coordinate system is used in describing the J_2 disturbance in the local (x - y - z) coordinate system.

This is depicted in Figure 1 and defined as the longitude of the ascending node, the argument of latitude, and the inclination angle, respectively, and is given by (Reference 7):

$$[ON] = \begin{bmatrix} \cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i & \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i & \sin\theta\sin i \\ -\cos\Omega\sin\theta - \sin\Omega\cos\theta\cos i & -\sin\Omega\sin\theta + \cos\Omega\cos\theta\cos i & \cos\theta\sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \quad (1a)$$

or

$$[ON] = \begin{bmatrix} e_{rX} & e_{rY} & e_{rZ} \\ e_{\theta X} & e_{\theta Y} & e_{\theta Z} \\ e_{hX} & e_{hY} & e_{hZ} \end{bmatrix} \quad (1b)$$

BASELINE HILL-CLOHESSY-WILTSHIRE EQUATION

Baseline Hill-Clohessy Wiltshire Equations, for circular orbit around the Earth as the central body, assumed the Earth as point mass centered at its center of mass and the center of the orbit.

The Hill-Clohessy Wiltshire Equations of motion in the chief LVLH frame are given by.

$$\frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} - 3\omega^2 x = 0 \quad (2a)$$

$$\frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} = 0 \quad (2b)$$

$$\frac{d^2z}{dt^2} + \omega^2 z = 0 \quad (2c)$$

which is also known as the Unperturbed HCW Equations. The angular velocity ω is given by

$$\omega = \sqrt{\frac{G(M+m)}{r^3}} = \sqrt{\frac{\mu}{r^3}} \tag{3}$$

The out-of-plane motion is modeled as a harmonic oscillator, where the in-plane motion is described as coupled harmonic oscillators. These second-order differential equations have the general solutions

$x(t) = A\cos(nt+\alpha)+x_{off}$ (4a) $y(t) = -2A\sin(nt+\alpha)-\frac{3}{2}nx_{off}t+y_{off}$ (4b) $z(t) = B\cos(nt+\beta)$ (4c)

where A, α, x_{off} , y_{off} ,B *and* β are the six integral constants. The velocities are found as the time derivatives of (6a,b,c). In order to produce bounded relative motion, the radial offset term must be equal to zero to eliminate the secular growth present in the alongtrack direction. For the z direction, integration of:

$$\delta z(t) = B_0 \sin(\omega t+\alpha) \tag{5}$$

yields $z(t) = -\frac{B_0}{\omega}\cos(\omega t+\alpha)+D_0$ or $z(t) = B\cos(nt+\alpha)+D_0$ (6)

Following Reference 11, the analytical solutions of the homogeneous CW equations are obtained as follows. Define $\mathbf{X} = [x \quad y \quad z]^T$ and $\mathbf{V} = [\dot{x} \quad \dot{y} \quad \dot{z}]^T$. A subscript 0 denotes the initial condition. Then the solution of the linearized Clohessy-Wiltshire (CW) equations can be represented in the following matrix form:

$$\mathbf{X}(t) = \Phi_{XX}(t)\mathbf{X}(t_0)+\Phi_{XV}(t)\mathbf{V}(t_0) \tag{7a}$$

$$\mathbf{V}(t) = \Phi_{VX}(t)\mathbf{X}(t_0)+\Phi_{VV}(t)\mathbf{V}(t_0) \tag{7b}$$

where $\Phi_{XX}(t)$, $\Phi_{XV}(t)$, $\Phi_{VX}(t)$ and $\Phi_{VV}(t)$ are state-transition matrices defined in Reference 4 and elaborated in Reference 11. The homogeneous solutions of the CW equation determine the position and the velocity of the deputy spacecraft relative to the chief spacecraft as a function of t subject to initial conditions $\boldsymbol{X}_0$ and $\boldsymbol{V}_0$.

RELATIVE BOUNDED MOTION

In formation flying, the motion of *deputy* satellite must remain bounded with respect to the chief satellite such that it experiences no secular drift and the formation configuration is maintained. One needs to find the condition such that the solutions of the Clohessy-Wiltshire equations are bounded (Reference 10).

Equation (2a) and Equation (2b) are coupled and they can be solved in parallel. Equation (2b) can be rewritten for $\dot{y}$ (t) :

$$\dot{y}(t) = -2\omega x(t)+2\omega x_0+\dot{y}_0 \tag{8}$$

If one integrate Equation (8) from 0 to t, one find terms that grow unboundedly over time, namely the terms $2\omega x_0(t)$ and $\dot{y}_0(t)$. However, $y(t)$ can be made bounded and periodic given the condition

$$2\omega x_0+\dot{y}_0 = 0 \tag{9}$$

Then, the solution for the in plane motion of the deputy satellite is:

$$x(t) = A_0 \sin(\omega t+\alpha) \tag{10a}$$

$$y(t) = 2A_0 \cos(\omega t+\alpha)+C_0 \tag{10b}$$

where A_0 , phase angle α and integration constant C_0 depend on the initial conditions. The out-of-plane motion is decoupled from the in-plane motion and its solution takes on the form of a simple harmonic oscillator:

$$z(t) = B_0 \sin(\omega t+\alpha) \tag{10c}$$

where amplitude B_0 and phase angle α are constants which depend on the initial conditions. The out-of-pane motion is periodic and bounded with respect to the chief satellite. The set of solutions in Equation (10) define a family of bounded, periodic motion trajectories for the deputy satellite in the relative frame under the assumptions of the HCW-equations. The motion of the deputy satellite, if projected onto the y-z plane, follows an ellipse of semi-major axis $2A_0$ and semi-minor axis A_0.

J_2 GRAVITATIONAL PERTURBATION EFFECTS

The previous section assumed the central body was a sphere of uniform density. This allows the two-body equations of motion to be written in a more simplified form. However, the Earth is not a perfect sphere with uniform density. Therefore, we would like to determine the gravitational potential due to an aspherical central body. In order to determine the gravitational potential at point P, each point in the Earth, Q m must be taken into account. The angles φ_{sat} and φ_Q are the respective colatitudes, λ_Q and θ_{sat} are the longitudinal arguments, and Λ is the angle between the vectors r_Q and r_{sat}, also known as the ground range or total range angle. All the above angle measurements are geocentric. The potential that describes an aspherical central body is then given (References 1, 2 and 3) as:

$$U = \frac{\mu}{r}\left[1 - \sum_{l=2}^{\infty} J_l \left(\frac{R_\oplus}{r}\right)^l P_l\left[\cos\left(\phi_{gc_{sat}}\right)\right] + \sum_{l=2}^{\infty}\sum_{m=1}^{l}\left(\frac{R_\oplus}{r}\right)^l P_{l,m}\left[\cos\left(\phi_{gc_{sat}}\right)\right]\left\{C_{l,m}\cos\left(m\lambda_{sat}\right) + S_{l,m}\sin\left(m\lambda_{sat}\right)\right\}\right] \tag{11}$$

where J_l , $C_{l,m}$, and , $S_{l,m}$ are gravitational coefficients and $R_\oplus$ is the equatorial radius of the Earth. The gravitational potential is

$$\mu = GM_e = 3.986005 \times 10^{14} m^3 / s^2 \tag{12}$$

which is the first term of the more general Earth's gravitational potential. *The first term is the two-body potential, whereas the second term is the potential due to zonal harmonics (J_{ll} terms, where m=0, and represent bands of latitude).* An aspherical body which only deviates from a perfect sphere due to zonal harmonics is axially symmetric about the Z-axis. The third term represents two other harmonics. The sectorial harmonics, where $l = m$, represent bands of longitude, and tesseral harmonics, where $l \neq m \neq 0$, represent tile-like regions of the Earth. The J_2 coefficient is about 1000 times larger than the next largest aspherical coefficient, and is therefore very important when describing the motion of a satellite around the Earth.

If the gravitational potential is considered to be due to spherical Earth, i.e. the first term of equation (11), then the gravitational acceleration is given by

$$\bar{g}\left(\bar{r}\right) = -\frac{\mu}{r^2}\bar{r} \tag{13}$$

If J_2 is included, the gravitational potential due to the J_2 disturbance can be obtained (from Reference 1 , 2 and 4) as

$$U_{zonal} = \frac{\mu}{r} J_2 \left(\frac{R_\oplus}{r}\right)^2 P_2\left[\cos\left(\phi_{gc_{sat}}\right)\right] \tag{13}$$

which can further be reduced to

$$U = -\frac{\mu}{\rho} + \frac{\mu R_e^2 J_2}{\rho^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \tag{14}$$

where $P_2\left[\cos\left(\phi_{gc_{sat}}\right)\right]$ is the associated Legendre polynomial of J_2 and the second zonal gravitational coefficient according to the JGM-2 model has been calculated as $J_2 = 1.082626925638815\times10^{-3}$. Similarly $J_3 = -2.5356\times10^{-6}$.

The acceleration due to J_2 in ***the ECI frame*** is then calculated as the gradient of the potential

$$\nabla U_{J_2} = \bar{J}_2 = -\frac{3\mu J_2 R_\oplus^2}{r^4}\begin{bmatrix}\left(\frac{1}{2} - \frac{3}{2}\sin^2 i \sin^2\theta\right) \\ \sin^2 i \sin\theta\cos\theta \\ \sin i \cos i \sin\theta\end{bmatrix} \tag{15}$$

The chief and deputy equations of motion can be rewritten in the inertial frame as

$$\ddot{\bar{r}}_c = -\frac{\mu}{r_c^3}\bar{r}_c + \bar{J}_{2_c} \tag{16}$$

$$\ddot{\bar{r}}_d = -\frac{\mu}{r_d^3}\bar{r}_d + \bar{J}_{2_d} \tag{17}$$

The acceleration due to J_2 in the LVLH frame may be calculated from the gradient in the r and Z directions:

$$\nabla U_{J_2} = \frac{\partial U_{J_2}}{\partial r}\hat{e}_r + \frac{\partial U_{J_2}}{\partial z}\hat{e}_z = -\mu J_2 R_\oplus^2\left[\left(\frac{3}{2r^4} - \frac{15z^3}{2r^6}\right)\hat{e}_r + \frac{3z}{r^5}\hat{e}_z\right] \tag{18}$$

The ($r - \theta - i$) coordinate system (Earth Centered Chief Satellite Orbital Plane coordinate system*)* is used in describing the J_2 disturbance in the local (x - y - z) coordinate system. The presence of r and the two Euler angles, θ and i , complete the geometry of the associated transform from the ECI frame to the ($r - \theta - i$) frame, utilizing the direction cosine matrix formed by the 3-1-3 Euler angle sets Ω , i and θ . This is defined as the longitude of ascending node, the argument of latitude, and the inclination angle, respectively.

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = r\begin{bmatrix} \cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i & \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i & \sin\theta\sin i \\ -\cos\Omega\sin\theta - \sin\Omega\cos\theta\cos i & -\sin\Omega\sin\theta + \cos\Omega\cos\theta\cos i & \cos\theta\sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix}\begin{Bmatrix} I_X \\ I_Y \\ I_Z \end{Bmatrix} \tag{19}$$

The acceleration due to J_2 in the LVLH frame may be calculated from the gradient in the r and Z directions:

$$\nabla U_{J_2} = \frac{\partial U_{J_2}}{\partial r}\hat{e}_r + \frac{\partial U_{J_2}}{\partial z}\hat{e}_z = -\mu J_2 R_\oplus^2\left[\left(\frac{3}{2r^4} - \frac{15z^3}{2r^6}\right)\hat{e}_r + \frac{3z}{r^5}\hat{e}_z\right] \tag{20}$$

where the Z component may be expressed in the LVLH frame as:

$$\hat{e}_z = \sin i\sin\theta\hat{e}_r + \sin i\cos\theta\hat{e}_r + \cos i\hat{e}_r \tag{21}$$

$$z = r\cos\phi = r\sin i\sin\theta \tag{22}$$

Substituting this back into Eq.(20) yields the acceleration due to J_2 gradient to be

$$\nabla U_{J_2} = \bar{J}_2 = -\frac{\mu J_2 R_\oplus^2}{r^4}\left[\left(\frac{1}{2} - \frac{3\sin^2 i\sin^2\theta}{2}\right)\hat{i} + \sin^2 i\sin\theta\cos\theta\hat{j} + \sin i\cos i\sin\theta\hat{k}\right] \tag{23a}$$

or

$$\nabla\bar{J}_2 = -\frac{6\mu J_2 R_\oplus^2}{r^5}\begin{bmatrix} \left(1 - 3\sin^2 i\sin^2\theta\right) & \sin^2 i\sin 2\theta & \sin 2i\sin\theta \\ \sin^2 i\sin 2\theta & -\frac{1}{2} - \sin^2 i\left(\frac{1}{2} - \frac{7}{4}\sin^2\theta\right) & \frac{\sin 2i\cos\theta}{4} \\ \sin 2i\sin\theta & \frac{\sin 2i\cos\theta}{4} & -\frac{3}{4} + \sin^2 i\left(\frac{1}{2} + \frac{5}{4}\sin^2\theta\right) \end{bmatrix}$$

(23b)

in Earth-Centered Inertial (ECI)frame of reference. The effects of J_2 are intrinsically determined from the aspherical central body.

J_2 MODIFIED HCW EQUATION

The chief and deputy equations of motion in the inertial frame due to J_2 in the ECI frame is given by equations (16) and (19) which appear as two-body Keplerian motion with added J_2 perturbations and is the exact or truth model. .

Similar to unperturbed HCW case, in LVLH, the solution of the equations of motion can be represented in the following matrix form, where appropriate terms have to be formulated:

$$\delta\mathbf{X}(t) = [\Phi_{XX}]\delta\mathbf{X}_0 + [\Phi_{XV}]\delta\mathbf{V}_0 \tag{24}$$

and

$$\delta\mathbf{V}(t) = [\Phi_{VX}]\delta\mathbf{X}_0 + [\Phi_{VV}]\delta\mathbf{V}_0 \tag{25}$$

With the present baseline formulation, the approach follows closely the linearized approach of Schweighart's (Reference 5) and Ginn's (Reference 7). Computational procedure and code is then developed.

The inertial relative position and velocity is defined as the position and velocity of the deputy relative to the chief.

$$\left[\bar{\rho}\right]_N = \bar{r}_d - \bar{r}_c \tag{27}$$

$$\left[\dot{\bar{\rho}}\right]_N = \dot{\bar{r}}_d - \dot{\bar{r}}_c \tag{28}$$

The relative position in the LVLH is calculated using the direction cosine matrix,[ON], defined by Eq. (1a) and Eq. (1b). Hence:

$$\left[\bar{\rho}\right]_O = \left[ON\right]\left[\bar{\rho}\right]_N \tag{29}$$

The relative velocity in the LVLH frame is defined using the *transport theorem* as

$$\left[\dot{\bar{\rho}}\right]_O = \left[\dot{\bar{\rho}}\right]_N - \bar{\omega}_{O/N} \times \left[\bar{\rho}\right]_O \tag{30}$$

where $\bar{\omega}_{O/N}$ is the rotation rate of the LVLH frame, which is the angular velocity of the chief orbit as stated before, and is defined as the angular momentum times the magnitude of the position squared:

$$\bar{\omega}_{O/N} = \frac{\bar{r}_c \times \dot{\bar{r}}_c}{r_c^{\,2}} \tag{31}$$

As the relative position and velocity is now fully defined from the exact nonlinear equations of motion in the ECI frame (Eqns. (27-31), then these are the governing “exact” model to be used as a basis for comparison.

The Equation of motion for linearized J2 modified HCW equation becomes (references 6 and 7):

$$\frac{d^2x}{dt^2} - 2nc\frac{dy}{dt} - \left(5c^2 - 2\right)n^2 x = -3n^2 J_2 \frac{R_\oplus^2}{r_c}\left(\frac{1}{2} - \frac{3}{2}\sin^2 i \sin^2 kt - \frac{1+3\cos 2i}{8}\right) \tag{32a}$$

$$\frac{d^2y}{dt^2} + 2nc\frac{dx}{dt} = -\frac{3}{2}n^2 J_2 \frac{R_\oplus^2}{r_c}\sin^2 i \sin 2kt \tag{32b}$$

$$\frac{d^2z}{dt^2} + \left(3c^2 - 2\right)n^2 z = 0 \tag{32c}$$

where

$$k = nc + \frac{3\sqrt{\mu}J_2 R_\oplus^2}{2r_c^{\frac{7}{2}}}\cos^2 i \tag{32d}$$

The equations for the calculation of the influence of J_2 on the linearized (HCW) orbit are summarized below

$$\begin{aligned} x(t) = {} & \left[\frac{5s+3}{s-1}x_{h0} + \frac{2\sqrt{1+s}}{n(s-1)}\dot{y}_{h0} + \frac{1}{4}\frac{A_{J2}\left(3k - 2n\sqrt{1-s}\right)\sin^2 i}{k\left(-n^2 + n^2 s + 4k^2\right)}\right]\cos\left(n\sqrt{1-s}t\right) \\ & -\frac{1}{4}\frac{A_{J2}\left(3k - 2n\sqrt{1-s}\right)\sin^2 i}{k\left(-n^2 + n^2 s + 4k^2\right)}\cos 2kt + \frac{\dot{x}_{h0}}{n\sqrt{1-s}}\sin\left(n\sqrt{1-st}\right) - \frac{4(s+1)}{s-1}x_{h0} - \frac{2\sqrt{1+s}}{n(s-1)}y_{h0} \end{aligned} \tag{33a}$$

$$y(t)=\left[\frac{2(5s+3)\sqrt{1+s}}{(1-s)^{\frac{3}{2}}}x_{h0}+\frac{4(1+s)}{n(1-s)^{\frac{3}{2}}}\dot{y}_{h0}+\frac{1}{2}\frac{A_{J2}\left(2ns-3k\sqrt{1+s}+2n\right)\sin^2 i}{k\sqrt{1-s}\left(-n^2+n^2s+4k^2\right)}\right]\sin\left(n\sqrt{1-st}\right)$$

$$-\frac{1}{8}\frac{A_{J2}\left(5n^2s+4k^2+3n^2-6nk\sqrt{1+s}\right)\sin^2 i}{k\left(-n^2+n^2s+4k^2\right)}\sin 2kt-\frac{2\sqrt{1+s}}{n(s-1)}\dot{x}_{h0}\cos\left(n\sqrt{1-st}\right) \tag{33b}$$

$$z(t)=z_{h0}\cos\left(n\sqrt{1+3st}\right)+\frac{\dot{z}_{h0}}{n\sqrt{1+3s}}\sin\left(n\sqrt{1+3st}\right) \tag{33c}$$

where

$$A_{J2}\equiv-3n^2J_2\frac{R_\oplus^2}{r_c} \tag{33d}$$.

These equations are incorporated in the MATLAB™ computational routine following the scheme depicted in Figure 2.

METHOD OF PERTURBATIONS

The following derivations are adapted from References 7 and 12, and add the effects of second-order differential gravity to the *HCW* equations using the method of perturbations, assuming that the J2 linearized modified HCW Eqns. (4.20) have solutions of the following form:

$$x=x_h+\varepsilon x_p \qquad \dot{x}=\dot{x}_h+\varepsilon\dot{x}_p \qquad \ddot{x}=\ddot{x}_h+\varepsilon\ddot{x}_p \tag{34a,b,c}$$

$$y=y_h+\varepsilon y_p \qquad \dot{y}=\dot{y}_h+\varepsilon\dot{y}_p \qquad \ddot{y}=\ddot{y}_h+\varepsilon\ddot{y}_p \tag{34e,f,g}$$

$$z=z_h+\varepsilon z_p \qquad \dot{z}=\dot{z}_h+\varepsilon\dot{z}_p \qquad \ddot{z}=\ddot{z}_h+\varepsilon\ddot{z}_p \tag{34h,i,j}$$

The subscript h refers to the solutions of the *HCW* equations and in this context will be termed the homogeneous solution; the subscript p refers to the correction due to the nonlinear gravitational terms and will be termed the perturbation solution. Define our perturbation parameter as

$$\varepsilon=\frac{3}{2}\frac{\mu}{r_c^4} \tag{35}$$

It is noted that this perturbation parameter is not dimensionless for comparison with References 5 and 6. The new nonlinear equations of motion are now

$$\ddot{x}-2n\dot{y}-3n^2x=\varepsilon\left[y^2+z^2-2x^2\right] \tag{36a}$$

$$\ddot{y}+2n\dot{x}=2\varepsilon xy \tag{36b}$$

$$\ddot{z}+n^2z=2\varepsilon xz \tag{36c}$$

Substituting Equation (4.21) into (4.23) we obtain

$$\ddot{x}_h-2n\dot{y}_h-3n^2x_h+\varepsilon\left(\ddot{x}_p-2n\dot{y}_p-3n^2x_p\right)=\varepsilon\left[\left(y_h+\varepsilon y_p\right)^2+\left(z_h+\varepsilon z_p\right)^2-2\left(x_h+\varepsilon x_p\right)^2\right] \tag{37a}$$

$$\ddot{y}_h+2n\dot{x}_h+\varepsilon\left(\ddot{y}_p+2n\dot{x}_p\right)=2\varepsilon\left(x_h+\varepsilon x_p\right)\left(y_h+\varepsilon y_p\right) \tag{37b}$$

$$\ddot{z}_h+n^2z_h+\varepsilon\left(\ddot{z}_p+n^2z_p\right)=2\varepsilon\left(x_h+\varepsilon x_p\right)\left(z_h+\varepsilon z_p\right) \tag{37c}$$

Since $\varepsilon \ll 1$, after series expansion and then dropping the higher order terms of ε, the J2 linearized modified HCW equations of motion for the second-order gravitational perturbation in perturbed form by Ginn becomes:

$$\ddot{x}_h-2n\dot{y}_h-3n^2x_h=\varepsilon\left(y_h{}^2+z_h{}^2-2x_h{}^2\right) \tag{38a}$$

$$\ddot{y}_h+2n\dot{x}_h=2\varepsilon x_hy_h \tag{38b}$$

$$\ddot{z}_h+n^2z_h=2\varepsilon x_hz_h \tag{38c}$$

The particular the solution to the HCW equations may be solved for by direct integration yielding an analytical model. The effects of second-order differential gravity may be added to the *J_2 -Modified HCW* equations using the method of perturbations

$$\bar{x} = \bar{x}_h + \varepsilon \bar{x}_p \tag{39}$$

where the homogeneous solution is the exact solution obtained from the *J_2 –Modified HCW* equations, and the perturbed solution is solved by the following linear, constant coefficient, differential equations:

$$\frac{d^2 x_p}{dt^2} - 2n_0 \frac{dy_p}{dt} - \alpha^2 {n_0}^2 x = {y_h}^2 + {z_h}^2 - 2{x_h}^2 \tag{40a}$$

$$\frac{d^2 y_p}{dt^2} + 2n_0 \frac{dx_p}{dt} = 2x_h y_h \tag{40b}$$

$$\frac{d^2 z_p}{dt^2} + \beta^2 {n_0}^2 z_p = 2x_h z_h \tag{40c}$$

J_2 and J_3 MODIFIED HCW EQUATION

Similar to the derivation of J_2 modified linearized HCW equation, the influence of J_2 and J_3 can be derived following the development below

$$\ddot{\bar{r}}_c = -\frac{\mu}{r_c^3}\bar{r}_c + \bar{J}_{2c} + \bar{J}_{3c} \tag{41a}$$

$$\ddot{\bar{r}}_d = -\frac{\mu}{r_d^3}\bar{r}_d + \bar{J}_{2d} + \bar{J}_{3d} \tag{41b}$$

The acceleration due to J_3 in the LVLH frame may be assumed to follow similar derivation for J_2 hence it can be calculated from the gradient in the *r* and *Z* directions:

$$\nabla U_{J_3} = \frac{\partial U_{J_3}}{\partial r}\hat{e}_r + \frac{\partial U_{J_3}}{\partial z}\hat{e}_z \tag{42a}$$

$$\nabla U_{J_2} = \bar{J}_{2_c}(\bar{r}) = -\frac{3}{2}\frac{J_2 \mu R_\oplus^2}{r^4}\begin{bmatrix} \left(1 - 3\sin^2 i \sin^2\theta\right)e_x \\ \left(2\sin^2 i \sin\theta\cos\theta\right)e_y \\ \left(2\sin i \cos i \sin\theta\right)e_z \end{bmatrix} \tag{42b}$$

These equations may be directly integrated to obtain the closed-form solution of the perturbation. When compared to the two different solutions that only take one of the disturbances into account, it is seen that this new combined solution yields better results when compared to the truth model, providing for periodically bounded solutions in all relative component directions. *It was expected that the addition of the J_2 and J_3 perturbation to the HCW equations would be to reduce the growth in the cross-track direction.* It was also expected that adding the perturbation associated with second-order differential gravity would improve upon the in-plane motion of the new solution, and is done so by bounding the errors from truth in the along-track direction.

The fidelity of this new perturbed J_2 - Modified HCW model may be improved upon to eliminate the unbounded growths in amplitude in the radial and cross-track direction, by taking into account the precession of the angular velocity vector. It is also believed that further analysis by adding this influence will provide for a stronger criterion for eliminating the secular growth in the along-track direction.

COMPARISON OF BASELINE CLOHESSY-WILTSHIRE MODEL OF TWIN-SATELLITE ORBITS WITH J2-PERTURBED ONES

To demonstrate the influence of J_2 on the linearized (HCW) orbit of the Twin Satellite Formation Flying Orbits, the J_2 perturbed linearized HCW equations orbits are compared with the baseline ones. The initial condition are those given in Table 1. The results are exhibited in Figs. 2 to 7..

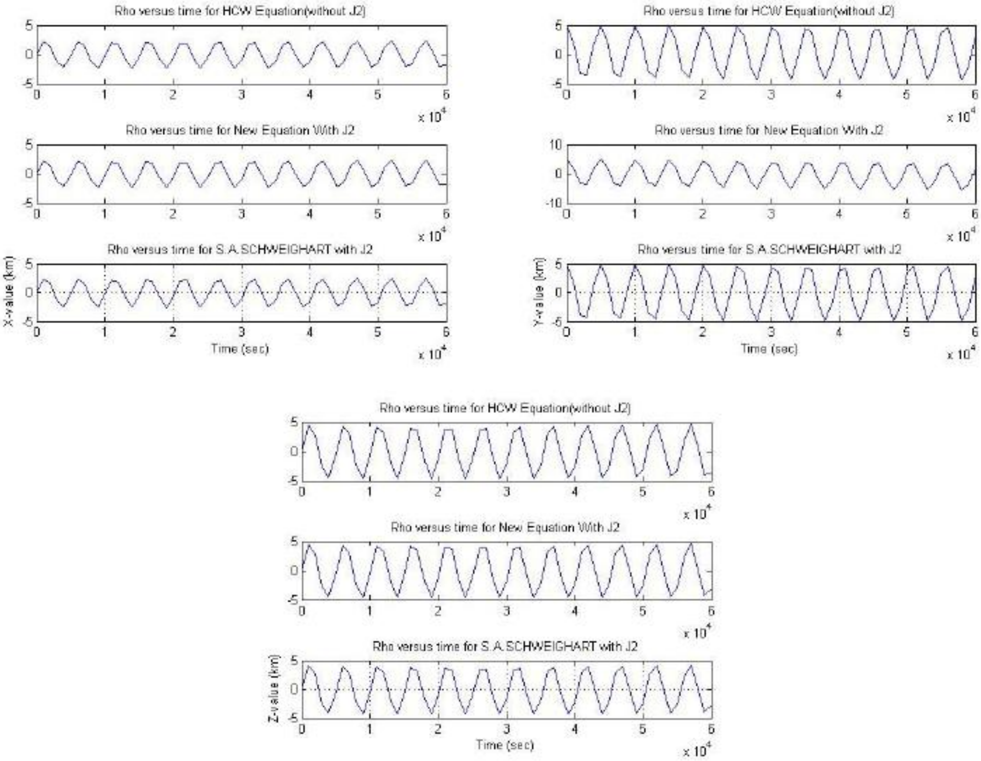

Figure 2: Comparison of the X,Y and Z values, respectively, of Deputy Satellite orbit around the Chief Satellite as the solution between baseline HCW, linearly J2 modified HCW Equation and Schweighart's results (the latter two incorporate the effect of J_2).

Table 1.
CHIEF'S ORBITAL ELEMENTS AND DEPUTY'S INITIAL CONDITIONS WITH RESPECT TO CHIEF'S

Chief Satellite	
Altitude, *h* (km)	847
Eccentricity, *e*	0
Orbit Inclination, *I* (deg)	10^0
Right Ascension of the Ascending Node, *Ω* (deg)	0^0
Argument of Perigee *ω* (deg)	0^0
Mean Anomaly at Epoch, *M* (deg)	0^0
Deputy Satellite Starting Condition (Chief-centered Frame)	
x_0 (km)	0.0
y_0 (km)	5.0
z_0 (km)	0.0
v_{x0} (km/s)	$0.5785 \cdot 10^{-3}$
v_{y0} (km/s)	0.0
v_{z0} (km/s)	$1.1570 \cdot 10^{-3}$
i (deg)	10
θ(deg)	nt

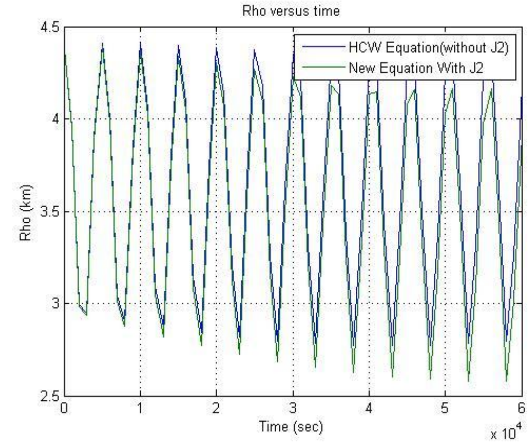

Figure 3. Comparison of the Deputy Satellite orbital radius around the Chief Satellite as the solution of Clohessy-Wiltshire Equation (without J_2) and incorporating the influence of J_2, using linearized modified Clohessy-Wiltshire Equation.

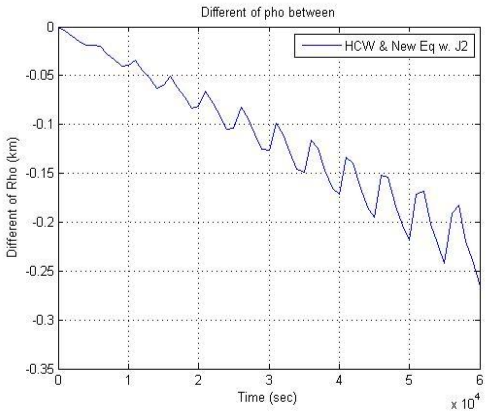

Figure 4. The difference between the radius of the orbit of the Deputy Satellite around the Chief Satellite as the solution of the original linearized HCW Equation (without J_2) and that incorporating the influence of J_2, using linearized J_2 modified HCW Equation

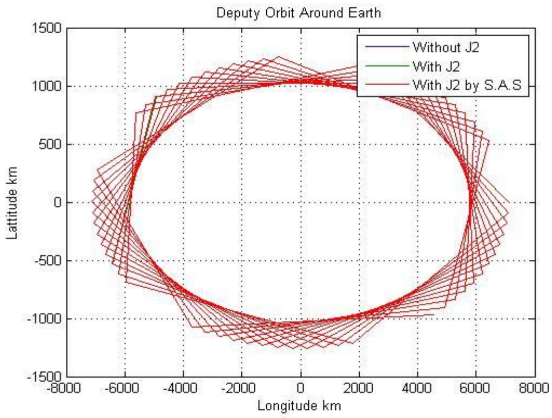

Figure 5. Comparison of Baseline Ground-Track of the Deputy and Chief Satellites orbits as the solution of baseline HCW and the Linearly Modified HCW equation which incorporate the influence of J_2.

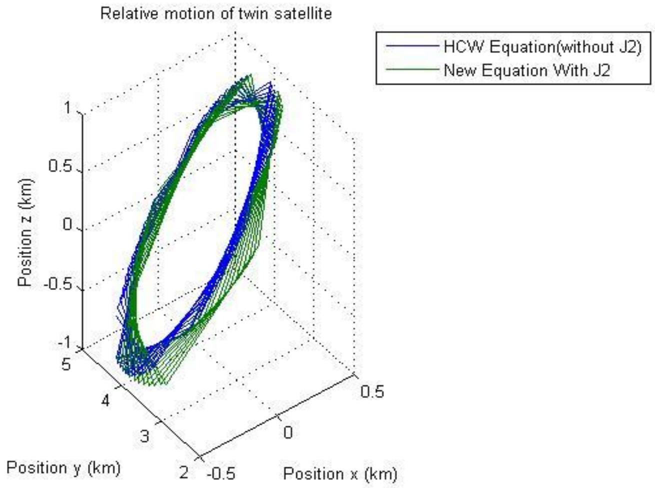

Figure 6. Comparison of Deputy Satellite orbit around the Chief Satellite as the solution of baseline Clohessy-Wiltshire Equations (without J_2) and Linearly Modified HCW equations which incorporate the influence of J_2.

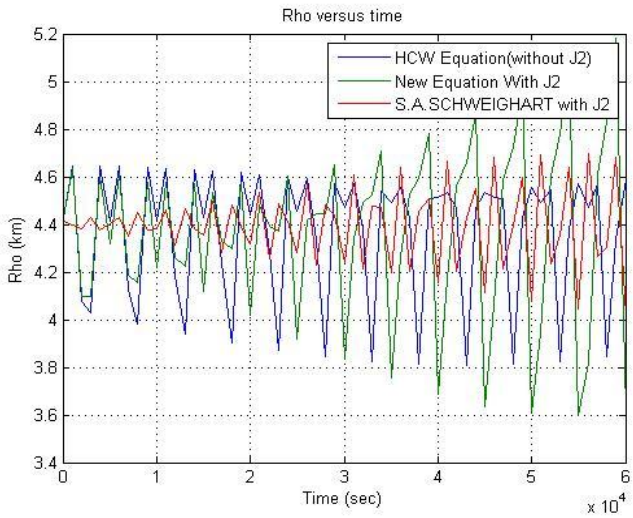

Figure 7: Comparison of the Deputy Satellite orbital radius around the Chief Satellite as the solution of the baseline Clohessy-Wiltshire Equations, the linearized J_2 modified HCW Equations and similar solution obtained by Schweighart [17].

The results show that the equations derived in this work have close similarity with the ones derived by Schweighart, although quantitatively there are differences. It should be noted that Schweighart's solutions originate from different J_2 linearization compared to the present work. Such difference may be attributed to the notion that the preset work does not include the drift of the ascending node of a satellite under the influence of the J_2 disturbance.

COMPARISON OF BASELINE CLOHESSY-WILTSHIRE MODEL OF TWIN-SATELLITE ORBITS WITH J2 AND J3-PERTURBED ONES

Similar to the derivation of the J2 Linearized Modified HCW Equations, the influence of J3 to the latter equation is derived using perturbation approximation. J3, which is $-2.5356\times10-6$, is about 1000 times smaller than J2, produces pear-shaped variations: a ~17 m bulge at North pole and ~7 m bulges at mid-southern latitudes. The detail is elaborated in a companion paper (Djojodihardjo, Reference 13).

$$U_{J_3} = -\frac{\mu J_3 R_\oplus{}^2}{r^5}\left[\frac{5Z^3}{2r^2} - \frac{3Z}{2}\right] = -\frac{\mu J_3 R_\oplus{}^2}{2}\left[\frac{5Z^3}{r^7} - \frac{3Z}{r^5}\right] \quad (43)$$

$$u = \frac{1}{2}\frac{J_3}{r^4}\sin\theta\left(5\sin^2\theta - 3\right) = \frac{z}{2}\frac{J_3}{r^7}\left(5z^2 - 3r^2\right) = \frac{J_3}{2}\left(\frac{5z^3}{r^7} - \frac{3z}{r^5}\right)$$

$$U_{J_3} = -\frac{\mu J_3 R_\oplus{}^2}{2}\left(\frac{5Z^3}{r^7} - \frac{3Z}{r^5}\right) \quad \textit{should be} \quad U_{J_3} = \frac{\mu J_3 R_\oplus{}^2}{r^3}\left(\frac{5}{3}\left(\frac{Z}{r}\right)^3 - \frac{3}{2}\left(\frac{Z}{r}\right)\right) \quad (44)$$

$$U_{J_3} = \frac{\mu J_3}{r}\left(\frac{R_\oplus}{r}\right)^3 P_3\left[\cos\left(\phi_{gc_{sat}}\right)\right] = \frac{\mu J_3}{r}\left(\frac{R_\oplus}{r}\right)^2\left(\frac{5}{3}\left(\cos\phi_{gc_{sat}}\right)^3 - \frac{3}{2}\left(\cos\phi_{gc_{sat}}\right)\right) \approx \frac{\mu J_3}{r}\left(\frac{R_\oplus}{r}\right)^2\left(\frac{5}{3}\left(\frac{z}{r}\right)^3 - \frac{3}{2}\left(\frac{z}{r}\right)\right) \quad (45)$$

$$\begin{aligned}\nabla U_{J_3} &= \frac{\partial U_{J_3}}{\partial r}\hat{e}_r + \frac{\partial U_{J_3}}{\partial z}\hat{e}_z \\ &= -\frac{\mu J_2 R_\oplus{}^2}{2}\left[\left(\frac{15Z}{r^6} - \frac{35Z^3}{r^8}\right)\hat{e}_r + \left(\frac{15Z^2}{r^7} - \frac{3}{r^5}\right)\hat{e}_z\right]\end{aligned} \quad (46)$$

In the following figures, the computational results obtained using J3 Linearized Modified HCW Equations are compared to the baseline HCW as well as J2 Linearized Modified HCW Equations results.

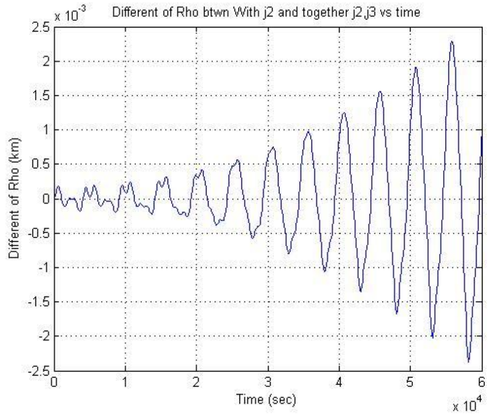

Figure 8 The difference in the radius of the orbit of the Deputy Satellite around the Chief Satellite between the solution of J2LMCW and the solution of J3LMCW.

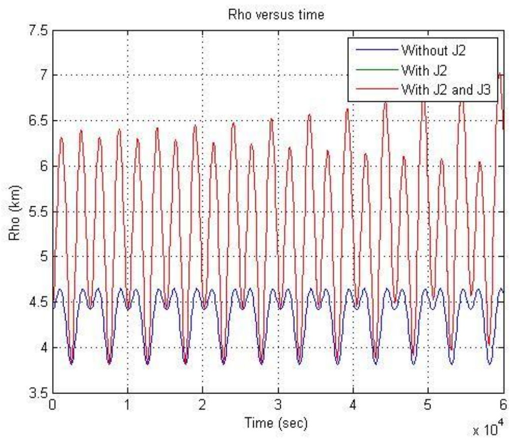

Figure 9. Comparison of the radius of the orbit of the Deputy Satellite around the Chief Satellite as the solution of CW, J2LMCW and J3LMCW.

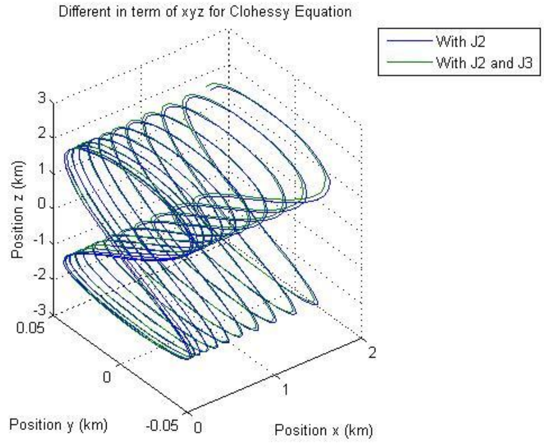

Figure 10. Comparison of the difference in the distance of the path of Deputy Satellite around the Chief Satellite in LVLH coordinate between the solution of CW Equation with J2LMCW equation and CW Equation with J3LMCW equation

CONCLUSION

Linearized Hill-Clohessy-Wiltshire equations have been utilized in developing modified form to take into account the influence of J_2 on the orbits of twin spacecraft in formation flight in near-Earth orbits. For Near Equatorial orbits the variation of J_2 is less apparent. Various relevant approaches and recent work on this issue have been synthesized into a novel and simplified approach, capitalizing on the balance between linearized approach and expected fidelity of the obtained solution, as stipulated by many earlier work. Judging from the accuracy estimation of simplified linearized approach, the exhibited computational results were obtained using J_2 linearized HCW equation. The original (baseline) linearized HCW approach and linearized J_2-modified HCW equation also exhibit the merit of simple analysis, which could be extended to incorporate other parameters. Similar analysis has been also carried out to obtain the influence of J3 on the formation flying of spacecrafts in Near Equatorial Low Earth orbits. The relevance of parametric study as a preliminary step towards optimization efforts has been demonstrated in the presentation of the results. The computation that has been performed using in-house developed MATLAB program. As a particular example, for near equatorial low earth orbit (i.e. 847 km), the error is about 0.25km from the desired relative position in the LVLH or Hill frame after 16.67 hours.

Acknowledgment

The author would like to thank Universiti Putra Malaysia (UPM) and Ministry of Higher Education, Malaysia for granting Research University Grant Scheme (RUGS) Project Code: 9378200 and Exploratory Research Grant Scheme (ERGS) ERGS 5527088, respectively, under which the present research is carried out.

References

1 Vinti, John P., (1971) Representation of the Earth's Gravitational Potential, MIT Measurement System Lab Report RE-71-NACA CR-115893.

2 Djojodihardjo, Harijono (1974.), Vinti's Surface Density As A Means Of Representing The Earth's Disturbance Potential, Proceedings ITB (ITB Journal), Vol.7, No. 4.

3 Anderson, D.L. (1989), Theory of the Earth, Ch.12, Blackwell Sci.Publ.

4 Alfriend, Kyle T. , Schaub, Hanspeter, Gim, Dong-Woo (2000), Gravitational Perturbations, Nonlinearity and Circular Orbit Assumption Effects on Formation Flying Control Strategies, AAS 00-012

5 Schweighart, S. A., "Development and Analysis of a High Fidelity Linearized J_2 Model for Satellite Formation Flying," M.S. thesis, Massachusetts Institute of Technology, 2001

[6] Schweighart and Sedwick Development and Analysis of a High Fidelity Linearized J2 Model for Satellite Formation Flying-AIAA 2001-4744

[7] Ginn, J.S. (2006), Spacecraft Formation Flight-Analysis of the Perturbed J2-Modified HCW Equation, MSc Thesis, University of Texas Arlington.

[8] Jennifer A Roberts, Satellite Formation Flying For an Interferometry Mission, PhD Thesis, Cranfield Institute of Technology, 2005

[9] Reid, T.; Misra, A.(2011), Formation flight of satellites in the presence of Atmospheric Drag, Journal of Aerospace Engineering, Sciences and Applications, Jan – April 2011, Vol. III, No 1 65.

[10] Alfriend K.T., Vadali S.R., Gurfil P., How J.P., Breger L.S. (2010), Spacecraft Formation Flying: Dynamics, Control and Navigation, Elsevier Astrodynamics Series, Burlington, USA.

[11] Vaddi, S.S., Vadali, S.R., and Alfriend, K.T., "Formation Flying: Accommodating Nonlinearity and Eccentricity Perturbations," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 2, 2003, pp. 214-223.

[12] Djojodihardjo, Harijono and Harithuddin, A.Salahuddin M.(2010), Spacecraft Formation Flying For Tropical Resources And Environmental Monitoring: A Parametric Study, AAS 10-441.

[13] Djojodihardjo, Harijono, (2014), Further Development of The Influence of the Earth's Gravitational Potential Perturbation on the Formation Flying Orbits of Micro-Satellites Near Equatorial Low Orbits, Internal Report, University Putra Malaysia, Aerospace Engineering Department.

[14] Lara, M and Gurfil, P., Integrable approximation of *J*2-perturbed relative orbits, Celest Mech Dyn Astr (2012) 114:229–254

[15] Martinusi, V. and Gurfil, P., Solutions and periodicity of satellite relative motion under even zonal harmonics perturbations