KINEMATIC EQUATIONS OF NONNOMINAL EULER AXIS/ANGLE ROTATION

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Euler axis/angle is a useful representation in many attitude control problems, being related to the single rotation that takes an "initial" reference frame to a "target" reference frame. As a matter of fact, there are some cases in which the nominal rotation cannot be performed. It is the case of spacecraft in underactuated conditions, where attitude effectors can deliver a control torque with two components only.

In a recent work an admissible rotation was proposed in order to minimize the alignment error between the target and the attainable attitude. In this paper, the kinematic equations of the nonnominal Euler axis/angle rotation are presented.*

INTRODUCTION

In attitude control problems Euler axis/angle representation is very useful to visualize the nominal rotation which takes an "initial" reference frame to a "target" reference frame by means of the minimum angular path.[1] In underactuated conditions, however, the nominal rotation is simply not attainable. Nonetheless a rotation around a nonnominal axis $\boldsymbol{g}$ can be performed, where a rotation is considered to be admissible if it takes place around an axis that lies on the plane orthogonal to the torqueless direction $\boldsymbol{b}$.

In Reference 2 one of the authors provided an exact analytical expression in order to compute the optimal rotation angle about a nonnominal rotation axis, which minimizes the alignment error between the target and the attainable attitude. In this case the admissible rotation axis $\boldsymbol{g}$ is derived from the nominal rotation axis $\boldsymbol{e}$ using the equation $\boldsymbol{g} = (\boldsymbol{b} \times \boldsymbol{e}/||\boldsymbol{b} \times \boldsymbol{e}||) \times \boldsymbol{b}$, and the rotation angle about $\boldsymbol{g}$ is the optimal one. More recently, Avanzini and Giulietti also demonstrated that it is always possible to determine a single admissible eigenaxis rotation that makes a single body-fixed axis parallel to a prescribed direction in space.[3] Summing up, these previous studies provided the two best options available with a single feasible eigenaxis rotation, either the minimum overall misalignment error or exact pointing of a single body-fixed axis.

In this paper the framework of kinematic planning of maneuvers in the presence of constraints on the direction of admissible rotation axes is completed. Kinematic equations, describing the evolution of nonnominal rotation axis/angle in the case when the overall misalignment error is minimized, are derived. A novel approach is presented allowing a generalized solution to the problem

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of underactuation when the torqueless direction is constant in either the target or the body frame. Examples of underactuated systems are represented by magnetically actuated spacecraft, where a desired torque cannot always be generated onboard. In particular, the body-referenced attainable torque is always perpendicular to the external magnetic field (whose direction is, of course, not controllable and variable in body frame) and to the onboard-generated magnetic dipole. In other words, the working principle of the attitude control hardware itself makes the system inherently underactuated, with the inability to provide three independent control torques at each time instant.[4] On the other hand, the spacecraft can become underactuated after a failure in a minimal control system or multiple failures in a redundant one, e.g. after the loss of a reaction wheel in a three-axes stabilized spacecraft with no redundancy.[5] In these cases, the application of well known control strategies is no longer possible for both regulation and tracking, and new methods have been proposed for tackling this particular problem.[6] To this aim, the present work addresses a framework of mathematical tools whose natural application is represented by the design of three-axis control laws in which the nonnominal axis $\boldsymbol{g}$ is the instantaneous rotation axis and the rotation angle to be used as the feedback term is the optimal one.

In what follows a brief overview about Euler axis/angle representation is provided in the Preliminaries Section, while the nonnominal rotation planning scheme is introduced in the next one. The derivation of kinematic equations for the nonnominal rotation is then performed in a unified framework for the two cases in which the torqueless direction $\boldsymbol{b}$ is constant in the body and the target frame, respectively. A Section of concluding remarks ends the paper.

PRELIMINARIES

Define two arbitrary Cartesian coordinate frames: the "initial" reference frame, $\mathbb{F}_1$, and the "target" reference frame, $\mathbb{F}_2$. Let $\mathbb{T}_{12} \in \mathbb{R}^{3\times3}$ represent the rotation matrix that allows for the transformation

$$\boldsymbol{v}_1 = \mathbb{T}_{12}\,\boldsymbol{v}_2 \tag{1}$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ represent a generic vector expressed in $\mathbb{F}_1$ and $\mathbb{F}_2$, respectively. Suppose that the desired attitude is achieved when $\mathbb{F}_1$ is aligned with the target frame $\mathbb{F}_2$. According to Euler's theorem, this can be obtained by a single rotation of the frame $\mathbb{F}_1$ about an axis referred to as the Euler axis (or rotation eigenaxis), whose components do not depend on the particular reference frame ($\mathbb{F}_1$ or $\mathbb{F}_2$). In what follows, all vector components will be expressed in $\mathbb{F}_1$, unless noted otherwise.

Let $\boldsymbol{e} \in \mathbb{R}^3$ represent the Euler axis unit vector, and let $\phi \in (0, \pi)$ represent the Euler angle of rotation about the Euler axis. Euler axis and angle can be expressed as a function of the rotation matrix as follows:[7]

$$\cos\phi = \frac{1}{2}\left[\mathrm{tr}\left(\mathbb{T}_{12}\right) - 1\right] \tag{2}$$

$$\boldsymbol{e}^{\times} = \frac{1}{2\,\sin\phi}\left(\mathbb{T}_{12}^T - \mathbb{T}_{12}\right) \tag{3}$$

where $\mathrm{tr}\left(\mathbb{T}_{12}\right)$ is the trace of $\mathbb{T}_{12}$ and $\boldsymbol{e}^{\times}$ is the skew symmetric cross-product operator

$$\boldsymbol{e}^{\times} = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} \tag{4}$$

On the converse, the reciprocal equations lead to an expression of the rotation matrix in terms of Euler axis/angle:

$$\mathbb{T}_{12} = \boldsymbol{I}_3 - \sin\phi\, \boldsymbol{e}^{\times} + (1 - \cos\phi)\ \boldsymbol{e}^{\times}\boldsymbol{e}^{\times} \tag{5}$$

provided $\boldsymbol{I}_3$ is the 3×3 unit matrix.

The kinematics of Euler axis/angle representation when the angular velocity is known is provided. Let $\boldsymbol{\omega}_{21} \in \mathbb{R}^3$ be the angular velocity of $\mathbb{F}_1$ relative to $\mathbb{F}_2$, expressed in $\mathbb{F}_1$. It is:[8]

$$\dot{\phi} = \boldsymbol{e} \cdot \boldsymbol{\omega}_{21} \tag{6}$$

where $\boldsymbol{e} \cdot \boldsymbol{\omega}_{21}$ is the scalar product between $\boldsymbol{e}$ and $\boldsymbol{\omega}_{21}$, and

$$\dot{\boldsymbol{e}} = \frac{1}{2}\left[\boldsymbol{e}^{\times} - \cot\left(\frac{\phi}{2}\right)\boldsymbol{e}^{\times}\boldsymbol{e}^{\times}\right]\boldsymbol{\omega}_{21} \tag{7}$$

PROBLEM STATEMENT

Given a generic attitude achievable by a rotation $\phi \neq 0, \pi$ about the eigenaxis $\boldsymbol{e}$ (the two cases $\phi = 0, \pi$ represent singularities in the Euler's theorem, see Eq. (3), and therefore must be excluded), there could be cases where the desired rotation $\mathcal{R}_{des}\left(\boldsymbol{e}, \phi\right)$ cannot be performed. Let $\boldsymbol{g} \in \mathbb{R}^3$ be a unit vector not aligned to the Euler axis $\boldsymbol{e}$: a rotation about $\boldsymbol{g}$ by the angle ϕ would take $\mathbb{F}_1$ to a frame $\mathbb{F}_3$ which is not aligned to the target frame $\mathbb{F}_2$.

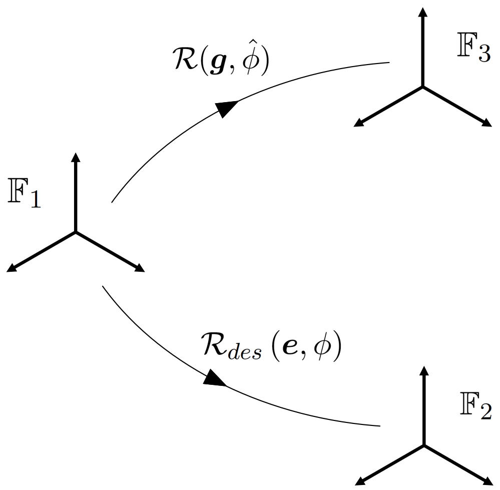

Figure 1. Definition of desired and admissible rotations

In Reference 2 an analytical expression was derived for the particular rotation angle $\hat{\phi} \in [0, \pi)$, about a generic axis $\boldsymbol{g}$ not aligned to $\boldsymbol{e}$, which minimizes the alignment error ϵ between the target frame $\mathbb{F}_2$ and the attainable frame $\mathbb{F}_3$ (see Figure 1). In particular, it was proven that[2,9]

$$\tan\left(\frac{\hat{\phi}}{2}\right) = (\boldsymbol{e} \cdot \boldsymbol{g}) \tan\left(\frac{\phi}{2}\right) \tag{8}$$

If, in addition, the admissible rotations $\mathcal{R}(\boldsymbol{g}, \hat{\phi})$ are constrained on a plane orthogonal to a unit vector $\boldsymbol{b} \in \mathbb{R}^3$, it was shown how the best admissible axis $\boldsymbol{g}$ is obtained when the angle between

e and g is minimum, thus leading to the minimum misalignment error, $\epsilon = \epsilon(\boldsymbol{b}, \mathbb{T}_{12})$. This latter condition is accomplished when $\boldsymbol{g}$ is parallel to the projection of $\boldsymbol{e}$ onto the plane of admissible rotations (see Figure 2), namely:

$$\boldsymbol{g} = \frac{\boldsymbol{b} \times \boldsymbol{e}}{\|\boldsymbol{b} \times \boldsymbol{e}\|} \times \boldsymbol{b} \tag{9}$$

and $\hat{\phi}$ is the optimal one defined in Eq. (8).

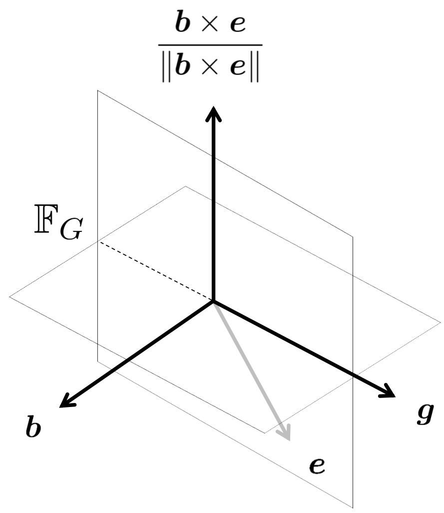

Figure 2. Definition of the ad hoc frame $\mathbb{F}_G$

As a final consideration, it was demonstrated that the minimum misalignment error $\epsilon \in [0, \pi)$ is given by:[9]

$$\epsilon = 2\cos^{-1}\left[\cos^2\left(\frac{\phi}{2}\right) + (\boldsymbol{e} \cdot \boldsymbol{g})^2 \sin^2\left(\frac{\phi}{2}\right)\right]^{1/2} \tag{10}$$

Taking into account Eq. (8), it follows:

$$\epsilon = 2\cos^{-1}\left[\cos\left(\frac{\phi}{2}\right) \Big/ \cos\left(\frac{\hat{\phi}}{2}\right)\right] \tag{11}$$

KINEMATIC EQUATIONS OF NONNOMINAL EULER AXIS/ANGLE

In this Section, kinematic equations describing the evolution of nonnominal axis/angle introduced in Eqs. (8) and (9) are derived. By taking into consideration Eq. (9), note that the three unit vectors $\boldsymbol{b}$, $\boldsymbol{g}$, and $(\boldsymbol{b} \times \boldsymbol{e})/||\boldsymbol{b} \times \boldsymbol{e}||$ form a right-handed triad $\mathbb{F}_G$ rotating with angular velocity $\boldsymbol{\omega}_{1G}$ with respect to $\mathbb{F}_1$. Since $\boldsymbol{g}$ is a constant in $\mathbb{F}_G$, the time derivative of the unit vector $\boldsymbol{g}$, calculated with respect to $\mathbb{F}_1$, can be written as

$$\dot{\boldsymbol{g}} = \boldsymbol{\omega}_{1G} \times \boldsymbol{g} \tag{12}$$

provided all the vector components are expressed in $\mathbb{F}_1$, unless noted otherwise.

In what follows, an explicit expression for the angular velocity $\boldsymbol{\omega}_{1G}$ will be derived for the two cases in which the torqueless direction $\boldsymbol{b}$ is prescribed in either $\mathbb{F}_1$ (Case 1) or $\mathbb{F}_2$ (Case 2).

Case 1

Suppose $\boldsymbol{b}$ is a constant in $\mathbb{F}_1$. The time derivative of the unit vector $(\boldsymbol{b}\times\boldsymbol{e})/||\boldsymbol{b}\times\boldsymbol{e}||$, expressed in $\mathbb{F}_1$, is given by (see Appendix 1)

$$\frac{d}{dt}\left(\frac{\boldsymbol{b}\times\boldsymbol{e}}{\|\boldsymbol{b}\times\boldsymbol{e}\|}\right)=\frac{(\boldsymbol{b}\times\boldsymbol{e})\times(\boldsymbol{b}\times\dot{\boldsymbol{e}})}{\|\boldsymbol{b}\times\boldsymbol{e}\|^2}\times\frac{\boldsymbol{b}\times\boldsymbol{e}}{\|\boldsymbol{b}\times\boldsymbol{e}\|} \tag{13}$$

where $\dot{\boldsymbol{e}}$ is calculated in Eq. (7). Note that the latter equation possesses the same structure of Eq. (12). By applying Poisson's Theorem for vector derivatives to the unit vector $(\boldsymbol{b}\times\boldsymbol{e})/||\boldsymbol{b}\times\boldsymbol{e}||$, it is possible to infer

$$\boldsymbol{\omega}_{1G}=\frac{(\boldsymbol{b}\times\boldsymbol{e})\times(\boldsymbol{b}\times\dot{\boldsymbol{e}})}{\|\boldsymbol{b}\times\boldsymbol{e}\|^2} \tag{14}$$

Consider now the definition of the optimal rotation angle in Eq. (8). Since $\boldsymbol{e}\cdot\dot{\boldsymbol{g}}=0$ (see Appendix 2), the time derivative of both sides of Eq. (8) provides:

$$\left[1+\tan^2\left(\frac{\hat{\phi}}{2}\right)\right]\frac{\dot{\hat{\phi}}}{2}=(\dot{\boldsymbol{e}}\cdot\boldsymbol{g})\tan\left(\frac{\phi}{2}\right)+(\boldsymbol{e}\cdot\boldsymbol{g})\left[1+\tan^2\left(\frac{\phi}{2}\right)\right]\frac{\dot{\phi}}{2} \tag{15}$$

that, upon substitution of the kinematics introduced in Eqs. (6) and (7), becomes

$$\dot{\hat{\phi}}=\frac{\tan\left(\frac{\phi}{2}\right)\left[\boldsymbol{e}^{\times}-\cot\left(\frac{\phi}{2}\right)\boldsymbol{e}^{\times}\boldsymbol{e}^{\times}\right]\boldsymbol{\omega}_{21}+\left[1+\tan^2\left(\frac{\phi}{2}\right)\right](\boldsymbol{e}\cdot\boldsymbol{\omega}_{21})\,\boldsymbol{e}}{1+\tan^2\left(\frac{\hat{\phi}}{2}\right)}\cdot\boldsymbol{g} \tag{16}$$

Taking into account that $(\boldsymbol{e}\cdot\boldsymbol{\omega}_{21})\,\boldsymbol{e}=(\boldsymbol{I}_3-\boldsymbol{e}^{\times}\boldsymbol{e}^{\times})\,\boldsymbol{\omega}_{21}$, Eq. (16) assumes the structure:

$$\dot{\hat{\phi}}=\frac{\left[1+\tan^2\left(\frac{\phi}{2}\right)\right]\left[2\boldsymbol{I}_3+\sin\phi\,\boldsymbol{e}^{\times}+(1-\cos\phi)\ \boldsymbol{e}^{\times}\boldsymbol{e}^{\times}\right]\boldsymbol{\omega}_{21}}{2\left[1+\tan^2\left(\frac{\hat{\phi}}{2}\right)\right]}\cdot\boldsymbol{g} \tag{17}$$

where it is easy to prove, from Eq. (5), that $\mathbb{T}_{12}^T=\boldsymbol{I}_3+\sin\phi\,\boldsymbol{e}^{\times}+(1-\cos\phi)\ \boldsymbol{e}^{\times}\boldsymbol{e}^{\times}$. By simple goniometric considerations, it follows:

$$\dot{\hat{\phi}}=\frac{\cos^2\left(\frac{\hat{\phi}}{2}\right)\left[\boldsymbol{I}_3+\mathbb{T}_{12}^T\right]}{2\cos^2\left(\frac{\phi}{2}\right)}\cdot\boldsymbol{g} \tag{18}$$

Consider now the expression of the final misalignment error ϵ in Eq. (11) and the goniometric relation $1/\cos(\epsilon/2)=\sec(\epsilon/2)$. Accordingly, the kinematics of nonnominal rotation angle $\hat{\phi}$ assumes the final compact form

$$\dot{\hat{\phi}}=\boldsymbol{\omega}_{21}^T\,\boldsymbol{\Phi}\boldsymbol{g} \tag{19}$$

provided

$$\boldsymbol{\Phi}(\boldsymbol{b},\mathbb{T}_{12})\triangleq\frac{1}{2}\sec^2\left(\frac{\epsilon}{2}\right)\left[\boldsymbol{I}_3+\mathbb{T}_{12}\right] \tag{20}$$

Case 2

Suppose $\boldsymbol{b}$ is a constant in $\mathbb{F}_2$. The time derivative of the unit vector $(\boldsymbol{b} \times \boldsymbol{e})/||\boldsymbol{b}\times\boldsymbol{e}||$, expressed in $\mathbb{F}_2$, assumes the same structure derived in Eq. (13) provided all the vector components are expressed in $\mathbb{F}_2$ (see Appendix 1). In other words:

$$\frac{d}{dt}\left(\frac{\boldsymbol{b}_2 \times \boldsymbol{e}}{\|\boldsymbol{b}_2 \times \boldsymbol{e}\|}\right) = \frac{(\boldsymbol{b}_2 \times \boldsymbol{e}) \times (\boldsymbol{b}_2 \times \dot{\boldsymbol{e}})}{\|\boldsymbol{b}_2 \times \boldsymbol{e}\|^2} \times \frac{\boldsymbol{b}_2 \times \boldsymbol{e}}{\|\boldsymbol{b}_2 \times \boldsymbol{e}\|} \tag{21}$$

provided $\boldsymbol{b}_2 = \mathbb{T}_{12}^T \boldsymbol{b}$, while it is straightforward to prove that the kinematics of Euler axis/angle, conveniently expressed as a function of $\boldsymbol{\omega}_{21}^{(2)}$, is given by:

$$\dot{\phi} = \boldsymbol{e} \cdot \boldsymbol{\omega}_{21}^{(2)} \tag{22}$$

$$\dot{\boldsymbol{e}} = -\frac{1}{2}\left[\boldsymbol{e}^{\times} + \cot\left(\frac{\phi}{2}\right)\boldsymbol{e}^{\times}\boldsymbol{e}^{\times}\right]\boldsymbol{\omega}_{21}^{(2)} \tag{23}$$

Note that in the Eqs. (21)-(23) the invariance of the components of $\boldsymbol{e}$ and $\dot{\boldsymbol{e}}$ between $\mathbb{F}_1$ and $\mathbb{F}_2$ has been taken into account. The right-hand side of Eq. (21) again resembles Poisson's Theorem applied to the unit vector $(\boldsymbol{b} \times \boldsymbol{e})/||\boldsymbol{b} \times \boldsymbol{e}||$, with components expressed in $\mathbb{F}_2$. Thereupon

$$\boldsymbol{\omega}_{2G}^{(2)} = \frac{(\boldsymbol{b}_2 \times \boldsymbol{e}) \times (\boldsymbol{b}_2 \times \dot{\boldsymbol{e}})}{\|\boldsymbol{b}_2 \times \boldsymbol{e}\|^2} \tag{24}$$

whose components can be rotated into the frame $\mathbb{F}_1$ through the transformation:

$$\boldsymbol{\omega}_{2G} = \mathbb{T}_{12}\,\boldsymbol{\omega}_{2G}^{(2)} = \frac{(\boldsymbol{b} \times \boldsymbol{e}) \times (\boldsymbol{b} \times \mathbb{T}_{12}\,\dot{\boldsymbol{e}})}{\|\boldsymbol{b} \times \boldsymbol{e}\|^2} \tag{25}$$

provided $\|\boldsymbol{b}_2 \times \boldsymbol{e}\| = \|\boldsymbol{b} \times \boldsymbol{e}\|$. The angular velocity $\boldsymbol{\omega}_{1G}$, expressed in $\mathbb{F}_1$, to be used in Eq. (12), is thus derived as the vector sum

$$\boldsymbol{\omega}_{1G} = \boldsymbol{\omega}_{2G} + \boldsymbol{\omega}_{12} = \boldsymbol{\omega}_{2G} - \boldsymbol{\omega}_{21} \tag{26}$$

where $\boldsymbol{\omega}_{12} = -\boldsymbol{\omega}_{21}$.

With regards to the time evolution of the optimal angle $\hat{\phi}$ in the case when $\boldsymbol{b}$ is a constant in $\mathbb{F}_2$, the procedure adopted in Eqs. (15)-(19) still holds if all the vector components are expressed in $\mathbb{F}_2$. In this case the derivative of both sides of Eq. (8) with respect to time provides:

$$\left[1 + \tan^2\left(\frac{\hat{\phi}}{2}\right)\right]\frac{\dot{\hat{\phi}}}{2} = (\dot{\boldsymbol{e}}_2 \cdot \boldsymbol{g}_2)\tan\left(\frac{\phi}{2}\right) + (\boldsymbol{e} \cdot \boldsymbol{g}_2)\left[1 + \tan^2\left(\frac{\phi}{2}\right)\right]\frac{\dot{\phi}}{2} \tag{27}$$

on condition that $\boldsymbol{e} \cdot \dot{\boldsymbol{g}}_2 = 0$ (see Appendix 2). Upon substitution of Eqs. (22) and (23), then Eq. (27) is solved to give

$$\dot{\hat{\phi}} = \frac{-\tan\left(\frac{\phi}{2}\right)\left[\boldsymbol{e}^{\times} + \cot\left(\frac{\phi}{2}\right)\boldsymbol{e}^{\times}\boldsymbol{e}^{\times}\right]\boldsymbol{\omega}_{21}^{(2)} + \left[1 + \tan^2\left(\frac{\phi}{2}\right)\right]\left(\boldsymbol{e} \cdot \boldsymbol{\omega}_{21}^{(2)}\right)\boldsymbol{e}}{1 + \tan^2\left(\frac{\hat{\phi}}{2}\right)} \cdot \boldsymbol{g}_2 \tag{28}$$

which, after some manipulations (see Case 1), turns to

$$\dot{\hat{\phi}} = \frac{\cos^2\left(\dfrac{\hat{\phi}}{2}\right)\left[2\boldsymbol{I}_3 - \sin\phi\,\boldsymbol{e}^{\times} + (1-\cos\phi)\ \boldsymbol{e}^{\times}\boldsymbol{e}^{\times}\right]\boldsymbol{\omega}_{21}^{(2)}}{2\cos^2\left(\dfrac{\phi}{2}\right)} \cdot \boldsymbol{g}_2 \tag{29}$$

Taking into account Eqs. (5) and (11), the equation above becomes

$$\dot{\hat{\phi}} = \frac{\left[\boldsymbol{I}_3 + \mathbb{T}_{12}\right]\boldsymbol{\omega}_{21}^{(2)}}{2\ \cos^2\left(\dfrac{\epsilon}{2}\right)} \cdot \boldsymbol{g}_2 \tag{30}$$

Given $\boldsymbol{\omega}_{21}^{(2)} = \mathbb{T}_{12}\,\boldsymbol{\omega}_{21}$ and projecting all the vector components of Eq. (30) in $\mathbb{F}_1$, the kinematics of $\hat{\phi}$ assumes the same expression derived in Eq. (18), provided $\boldsymbol{\Phi}$ is the operator:

$$\boldsymbol{\Phi}(\boldsymbol{b}, \mathbb{T}_{12}) \triangleq \frac{1}{2}\sec^2\left(\frac{\epsilon}{2}\right)\ \left[\boldsymbol{I}_3 + \mathbb{T}_{12}^T\right] \tag{31}$$

Remark. The kinematic equations derived for the nonnominal angle $\hat{\phi}$ in Eqs.(19) and (20), and the analogous obtained for Case 2, actually represent a generalization of the nominal case in Eq. (6). In fact, when the nominal rotation axis $\boldsymbol{e}$ falls on the plane of admissible rotations, namely $\boldsymbol{g} \equiv \boldsymbol{e}$, then the attainable frame $\mathbb{F}_3$ coincides with $\mathbb{F}_2$, with a null residual error, $\epsilon = 0$. In this case, it is $\hat{\phi} \equiv \phi$ and $\dot{\hat{\phi}} \equiv \dot{\phi}$.

CONCLUSIONS

Euler axis/angle is a widely used attitude representation technique. In the case of underactuated spacecraft, however, admissible rotation axes are constrained on a plane perpendicular to the direction along which no control torque is available. Assuming that a reference frame $\mathbb{F}_1$ can be rotated about an Euler axis $\boldsymbol{e}$ by the Euler angle ϕ in order to reach the target reference frame $\mathbb{F}_2$, an analytical method was presented in a previous work to compute the rotation angle $\hat{\phi}$ by which $\mathbb{F}_1$ must be rotated about a nonnominal Euler axis $\boldsymbol{g}$ in order to minimize the attitude error between the attained reference frame $\mathbb{F}_3$ and the target frame $\mathbb{F}_2$. In the present paper, the framework of kinematic planning of maneuvers in the presence of constraints on the direction of admissible rotation axes was completed by the derivation of kinematic equations, describing the evolution of nonnominal rotation axis/angle with respect to time. A generalized solution is provided in the cases when the torqueless axis is prescribed in both $\mathbb{F}_1$ and $\mathbb{F}_2$. Examples of these two cases are represented by 1) underactuated spacecraft, owing to the possible lack of actuators capable of rotations around a body-fixed axis, and 2) magnetically controlled spacecraft, because of the variable direction, in body axes, of the geomagnetic field. The proposed approach proves to be simple from a mathematical standpoint and the kinematic equations, derived in a compact form, highly suit the framework of attitude control problems where a desired torque is not attainable.

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APPENDIX 1

Let $\boldsymbol{b} \in \mathbb{R}^3$ and $\boldsymbol{e} \in \mathbb{R}^3$ be unit vectors and suppose $\boldsymbol{b}$ is fixed with respect to the generic frame $\mathbb{F}_i$ (either $\mathbb{F}_1$ or $\mathbb{F}_2$ in this framework). The time derivative of the unit vector $(\boldsymbol{b} \times \boldsymbol{e})\,/||\boldsymbol{b} \times \boldsymbol{e}||$ with respect to $\mathbb{F}_i$ can be calculated as

$$\frac{d}{dt}\left(\frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|}\right) = \frac{(\boldsymbol{b}_i \times \dot{\boldsymbol{e}}_i)\ \|\boldsymbol{b}_i \times \boldsymbol{e}_i\| + (\boldsymbol{b}_i \times \boldsymbol{e}_i)\ \frac{d}{dt}\left(\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|\right)}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|^2} \tag{32}$$

provided all vector components are expressed in $\mathbb{F}_i$. Consider now the definition of Euclidean norm of a vector $\boldsymbol{v}$ in terms of inner product, namely

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} \tag{33}$$

Taking the derivative of both sides of Eq. (33), it is trivial to obtain:

$$\frac{d}{dt}\left(\|\boldsymbol{v}\|\right) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \dot{\boldsymbol{v}} \tag{34}$$

The formula above can be conveniently applied to the norm $\|\boldsymbol{b} \times \boldsymbol{e}\|$. It follows:

$$\frac{d}{dt}\left(\frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|}\right) = \frac{\boldsymbol{b}_i \times \dot{\boldsymbol{e}}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|} + \left(\frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|} \cdot \frac{\boldsymbol{b}_i \times \dot{\boldsymbol{e}}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|}\right) \frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|} \tag{35}$$

Note that the right-hand side of Eq. (35) can be recast in a more compact form according to the vector triple product rule, $(\boldsymbol{\alpha} \cdot \boldsymbol{\gamma})\ \boldsymbol{\beta} - (\boldsymbol{\alpha} \cdot \boldsymbol{\beta})\ \boldsymbol{\gamma} = \boldsymbol{\alpha} \times (\boldsymbol{\beta} \times \boldsymbol{\gamma})$, provided

$$\boldsymbol{\alpha} = \boldsymbol{\gamma} = \frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|} \qquad \boldsymbol{\beta} = \frac{\boldsymbol{b}_i \times \dot{\boldsymbol{e}}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|}$$

Accordingly:

$$\frac{d}{dt}\left(\frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|}\right) = \frac{(\boldsymbol{b}_i \times \boldsymbol{e}_i) \times (\boldsymbol{b}_i \times \dot{\boldsymbol{e}}_i)}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|^2} \times \frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|} \tag{36}$$

APPENDIX 2

Consider now the definition of nonnominal Euler axis in Eq. (9). The time derivative of $\boldsymbol{g}$ expressed in the generic frame $\mathbb{F}_i$ is

$$\dot{\boldsymbol{g}}_i = \frac{d}{dt}\left(\frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|} \times \boldsymbol{b}_i\right) \tag{37}$$

By hypothesis $\boldsymbol{b}$ is a constant in $\mathbb{F}_i$, and it can be carried out of the derivative in the equation above. It is

$$\dot{\boldsymbol{g}}_i = \frac{d}{dt}\left(\frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|}\right) \times \boldsymbol{b}_i \tag{38}$$

where the first term on the right-hand side is provided by Eq. (36). Hence

$$\dot{\boldsymbol{g}}_i = \left[\frac{(\boldsymbol{b}_i \times \boldsymbol{e}_i) \times (\boldsymbol{b}_i \times \dot{\boldsymbol{e}}_i)}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|^2} \times \frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|}\right] \times \boldsymbol{b}_i \tag{39}$$

where the double cross product can be manipulated to provide

$$\dot{\boldsymbol{g}}_i = \left[\frac{(\boldsymbol{b}_i \times \boldsymbol{e}_i) \times (\boldsymbol{b}_i \times \dot{\boldsymbol{e}}_i)}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|^2} \cdot \boldsymbol{b}_i\right] \frac{\boldsymbol{b}_i \times \boldsymbol{e}_i}{\|\boldsymbol{b}_i \times \boldsymbol{e}_i\|} \tag{40}$$

The equation above shows that vector $\dot{\boldsymbol{g}}$ aims in the direction of $(\boldsymbol{b} \times \boldsymbol{e})/||\boldsymbol{b} \times \boldsymbol{e}||$.