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ANALYTIC MODEL FOR THE MOTION ABOUT AN OBLATE PLANET IN THE PRESENCE ATMOSPHERIC DRAG

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The paper introduces a new model for the motion about an oblate planet under the influence of the atmospheric drag. Both qualitative and quantitative insights are revealed, as well as closed-form equations of motion. The main tool consists in averaging the effects of both perturbations (oblateness and drag), and deriving the variational equations for the vectorial orbital elements. The model is singularity-free and may serve as an initial guess for control problems, as well as an analytic propagator.

INTRODUCTION

The most significant non-conservative force acting upon satellites orbiting below the altitude of 700 kilometers is the atmospheric drag, which causes the orbit to drift very quickly from the assumed unperturbed Keplerian orbit. Combined with the dominant gravitational perturbation, namely the second zonal harmonic J_2, the orbit drifts even more from the nominal unperturbed model.

Finding analytic models for such orbits lacks the classic tools of Analytical Mechanics, since the forces involved are non-conservative. The first attempt to approach the combined effect of the two aforementioned perturbations was made by Brouwer and Hori,[1] by using the same tools as in the work introducing the classic Brouwer theory.[2] The resulting model is extremely laborious and difficult to implement, and it was not developed further. An analytic solution to the problem of the motion about an oblate planet with drag was developed by Mittleman and Jezewski,[3] but for the particular assumption of a drag force inversely proportional to the cubic of the radial distance. Although very elegant, this model does not appear to have any practical relevance. Another analytic approach to the effect of the atmospheric drag was made by Vinh and Longuski,[4] under the simplifying assumption of a relatively small eccentricity and by using Poincaré's integration method of small parameters, but without taking into account the effect of the oblateness. More recent results were obtained by Xu and Xu,[5] which derived analytic expressions for the orbital elements by using Bessel series expansions.

The present work uses a straightforward averaging method which was proposed by Hestenes[6] and was intended only for the effect of the J_2 zonal harmonic. The effect of the atmospheric drag is taken into account under the assumption of a constant atmospheric density. Time-explicit expressions for

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the classical orbital elements are obtained for two situations: small orbital eccentricity ($e^4 \simeq 0$) and very small eccentricity ($e^2 \simeq 0$).

PROBLEM FORMULATION

The motion of a satellite about an oblate planet in the presence of atmospheric drag is modeled by the initial value problem (IVP):[7]

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{a}_d,\ \ \mathbf{r}\left(t_0\right) = \mathbf{r}_0,\ \dot{\mathbf{r}}\left(t_0\right) = \mathbf{v}_0,\ \mathbf{a}_d = \mathbf{a}_{J_2} + \mathbf{a}_{drag} \tag{1}$$

$$\mathbf{a}_{J_2} = -\frac{3\mu J_2 r_{eq}^2}{2r^4}\left[\left(1 - 5\cos^2\phi\right)\frac{\mathbf{r}}{r} + 2\cos\phi\,\mathbf{i}_z\right];\ \ \mathbf{a}_{drag} = -\frac{1}{2}C_D\frac{S_{ref}}{m}\rho_0\left\|\mathbf{v}\right\|\mathbf{v}. \tag{2}$$

Here μ is the gravitational parameter, J_2 is the second zonal harmonic, ϕ the colatitude angle, $\mathbf{i}_z$ the unit vector associated with the Earth rotation axis, r_{eq} the mean equatorial radius, C_D the drag coefficient, S_{ref} the cross-sectional area of the satellite, m its mass and ρ_0 the atmospheric density. The assumptions of a static uniform atmosphere, a constant drag coefficient and the presence of drag exclusively (no lift) are made.

THE VARIATIONAL EQUATIONS

Consider the unperturbed problem associated with Eq. (1), namely the IVP for $\mathbf{a}_d = \mathbf{0}$. Then the motion becomes Keplerian, and the following quantities represent constants of the motion:

$$\begin{cases} \mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \\ \mathbf{e} = \dfrac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} - \dfrac{\mathbf{r}}{r} \\ \mathcal{E} = \dfrac{1}{2}\dot{\mathbf{r}}^2 - \dfrac{\mu}{r} \end{cases} \tag{3}$$

which represent the specific angular momentum, the eccentricity vector and the specific total energy, respectively. If the specific energy $\mathcal{E}$ is negative, then the unperturbed orbit is elliptical, and its semimajor axis is defined as:

$$a = \frac{\mu}{2\left|\mathcal{E}\right|} \tag{4}$$

If the perturbed problem is now considered, then the variations of the elements defined in Eq. (3) are derived as:

$$\begin{cases} \dot{\mathbf{h}} = \mathbf{r} \times \mathbf{a}_d \\ \dot{\mathbf{e}} = \dfrac{1}{\mu}\left[\mathbf{a}_d \times \mathbf{h} - \left(\mathbf{r} \times \mathbf{a}_d\right) \times \dot{\mathbf{r}}\right] \\ \dot{a} = \dfrac{2a^2}{\mu}\left(\dot{\mathbf{r}} \cdot \mathbf{a}_d\right) \end{cases} \tag{5}$$

The averaged variations of the orbital elements $\mathbf{h}$, $\mathbf{e}$ and $\mathcal{E}$ are defined as:

$$\begin{cases} \dot{\overline{\mathbf{h}}} = \dfrac{1}{T}\int\limits_0^T (\mathbf{r} \times \mathbf{a}_d)\, \mathrm{d}t \\ \dot{\overline{\mathbf{e}}} = \dfrac{1}{\mu T}\int\limits_0^T [\mathbf{a}_d \times \mathbf{h} - (\mathbf{r} \times \mathbf{a}_d) \times \dot{\mathbf{r}}]\, \mathrm{d}t \\ \dot{\overline{a}} = \dfrac{2}{\mu T}\int\limits_0^T a^2 (\dot{\mathbf{r}} \cdot \mathbf{a}_d)\, \mathrm{d}t \end{cases} \tag{6}$$

where T is the period of the Keplerian unperturbed motion,

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{n} \tag{7}$$

with n being the mean motion. All the computations will be made with respect to the averaged perifocal frame, defined as in Figure 1:

$$\overline{\mathbf{u}}_1 = \frac{\overline{\mathbf{e}}}{\|\overline{\mathbf{e}}\|}; \quad \overline{\mathbf{u}}_2 = \frac{\overline{\mathbf{h}} \times \overline{\mathbf{e}}}{\|\overline{\mathbf{h}}\| \, \|\overline{\mathbf{e}}\|}; \overline{\mathbf{u}}_3 = \frac{\overline{\mathbf{h}}}{\|\overline{\mathbf{h}}\|}. \tag{8}$$

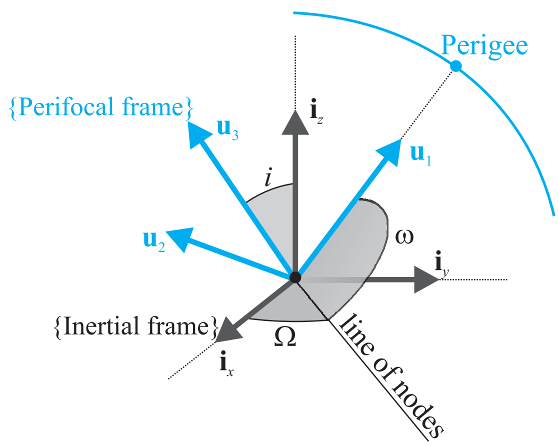

Figure 1. Averaged perifocal frame and the inertial Earth centered frame.

By taking into account the expression in Eq. (2) of the perturbing acceleration $\mathbf{a}_d$, after some computations, the averaged variational equations (6) become:

$$\begin{cases} \dot{\overline{\mathbf{h}}} = \dot{\overline{\mathbf{h}}}_{J_2} + \dot{\overline{\mathbf{h}}}_{drag} \\ \dot{\overline{\mathbf{e}}} = \dot{\overline{\mathbf{e}}}_{J_2} + \dot{\overline{\mathbf{e}}}_{drag} \\ \dot{\overline{a}} = \dot{\overline{a}}_{J_2} + \dot{\overline{a}}_{drag} \end{cases} \tag{9}$$

where:[8]

$$
\begin{cases}
\dot{\bar{\mathbf{h}}}_{J_2} = -\dfrac{3\mu J_2 r_{eq}^2}{2\bar{a}^3\left(1-\bar{e}^2\right)^{3/2}}\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)\left(\mathbf{i}_z\times\bar{\mathbf{u}}_3\right) \\[2ex]
\dot{\bar{\mathbf{e}}}_{J_2} = \dfrac{3\bar{e}\sqrt{\mu}J_2 r_{eq}^2}{2\bar{a}^{7/2}\left(1-\bar{e}^2\right)^2}\left\{-\left[1-3\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)^2\right]\bar{\mathbf{u}}_2+2\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_2\right)\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)\bar{\mathbf{u}}_3\right\} \\[2ex]
\dot{\bar{a}}_{J_2} = 0
\end{cases}
\tag{10}
$$

$$
\begin{cases}
\dot{\bar{\mathbf{h}}}_{drag} = -\dfrac{2C_0\mu\sqrt{1-\bar{e}^2}}{\pi}E\left(\bar{e}\right)\bar{\mathbf{u}}_3 \\[2ex]
\dot{\bar{\mathbf{e}}}_{drag} = -\dfrac{4C_0\sqrt{\mu}\left(1-\bar{e}^2\right)}{\pi\bar{e}\sqrt{\bar{a}}}\left[K\left(\bar{e}\right)-E\left(\bar{e}\right)\right]\bar{\mathbf{u}}_1 \\[2ex]
\dot{\bar{a}}_{drag} = -\dfrac{4C_0\sqrt{\mu\bar{a}}}{\pi}\left[2K\left(\bar{e}\right)-E\left(\bar{e}\right)\right]
\end{cases}
\tag{11}
$$

where C_0 is defined as:

$$
C_0 = \frac{1}{2}C_D\frac{S_{ref}}{m}\rho_0 \tag{12}
$$

and $K\left(\cdot\right)$ and $E\left(\cdot\right)$ are the complete elliptic integrals of the first and second kind, respectively, defined as:[9]

$$
K\left(k\right)=\int_0^1\frac{\mathrm{d}u}{\sqrt{1-k^2u^2}};\quad E\left(k\right)=\int_0^1\frac{\sqrt{1-u^2}\mathrm{d}u}{\sqrt{1-k^2u^2}},\quad\left(\forall\right)0\le k\le 1 \tag{13}
$$

A first qualitative insight on the averaged motion is revealed as follows: the time derivatives of the unit vectors $\bar{\mathbf{u}}_{1,2,3}$ are computed as follows:

$$
\begin{cases}
\dot{\bar{\mathbf{u}}}_1 = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\bar{\mathbf{e}}}{\bar{e}} = \dfrac{\bar{\mathbf{u}}_1\times\left(\dot{\bar{\mathbf{e}}}\times\bar{\mathbf{u}}_1\right)}{\bar{e}} \\[2ex]
\dot{\bar{\mathbf{u}}}_2 = \bar{\mathbf{u}}_3\times\dot{\bar{\mathbf{u}}}_1+\dot{\bar{\mathbf{u}}}_3\times\bar{\mathbf{u}}_1 \\[2ex]
\dot{\bar{\mathbf{u}}}_3 = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\bar{\mathbf{h}}}{\bar{h}} = \dfrac{\bar{\mathbf{u}}_3\times\left(\dot{\bar{\mathbf{h}}}\times\bar{\mathbf{u}}_3\right)}{\bar{h}}
\end{cases}
\tag{14}
$$

and based on Eqs. (9)–(11), Eqs. (14) become:

$$
\begin{cases}
\dot{\bar{\mathbf{u}}}_1 = C_u\left\{\left[3\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)^2-1\right]\bar{\mathbf{u}}_2+2\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_2\right)\bar{\mathbf{u}}_3\right\} \\[2ex]
\dot{\bar{\mathbf{u}}}_2 = -C_u\left[3\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)^2-1\right]\bar{\mathbf{u}}_1-2\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_1\right)\bar{\mathbf{u}}_3 \\[2ex]
\dot{\bar{\mathbf{u}}}_3 = C_u\left[-2\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_2\right)\bar{\mathbf{u}}_1+2\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_3\right)\left(\mathbf{i}_z\cdot\bar{\mathbf{u}}_1\right)\bar{\mathbf{u}}_2\right]
\end{cases}
\tag{15}
$$

where C_u is defined as:

$$C_u = \frac{3}{4} \frac{J_2 \sqrt{\mu} r_{eq}^2}{\bar{a}^{7/2} \left(1 - \bar{e}^2\right)^2} \tag{16}$$

As follows from the third of Eqs. (15):

$$\bar{\dot{\mathbf{u}}}_3 \cdot \mathbf{i}_z = 0 \tag{17}$$

and consequently the averaged inclination $\bar{i}$ of the orbit remains constant, since:

$$\cos \bar{i} = \bar{\mathbf{u}}_3 \cdot \mathbf{i}_z = \text{constant} \tag{18}$$

Regularization of The Variational Equations

The effect of J_2 zonal harmonic, as expressed in Eqs (10), leads to the well-known averaged expressions for the derivatives of the orbital elements:[8, 10, 11]

$$\begin{cases} \bar{\dot{\Omega}} = \dfrac{C_\Omega}{\bar{a}^{7/2} \left(1 - \bar{e}^2\right)^2}; & \overline{\Omega}\left(t_0\right) = \overline{\Omega}_0 \\ \\ \bar{\dot{\omega}} = \dfrac{C_\omega}{\bar{a}^{7/2} \left(1 - \bar{e}^2\right)^2}; & \overline{\omega}\left(t_0\right) = \overline{\omega}_0 \end{cases} \tag{19}$$

The averaged derivative of the mean anomaly M may be determined from the Gauss variational equations, and it has the expression:[10, 11]

$$\overline{\dot{M}} = \frac{\sqrt{\mu}}{\bar{a}^{3/2}} + \frac{C_M}{\bar{a}^{7/2} \left(1 - \bar{e}^2\right)^{3/2}} \tag{20}$$

The following notations were used:

$$\begin{cases} C_\Omega = -\dfrac{3}{2} J_2 \sqrt{\mu} r_{eq}^2 \\ \\ C_\omega = \dfrac{3}{4} J_2 \sqrt{\mu} r_{eq}^2 \left(5 \cos^2 \bar{i} - 1\right) \\ \\ C_M = \dfrac{3}{4} J_2 \sqrt{\mu} r_{eq}^2 \left(3 \cos^2 \bar{i} - 1\right) \end{cases} \tag{21}$$

Introduce the new independent variable τ such that:

$$\mathrm{d}t = \frac{-2}{C_0 \sqrt{\mu \bar{a}} \left(4 + 7\bar{e}^2\right) \left(1 - \bar{e}^2\right)} \mathrm{d}\tau, \; \tau\left(t_0\right) = \bar{a}_0, \tag{22}$$

where $\bar{a}_0$ is the value of the averaged semimajor axis at the initial moment of time $t = t_0$. Define the new differential operator $(\cdot)'$ such that:

$$(\cdot)' = \frac{d}{d\tau}(\cdot) \tag{23}$$

The atmospheric drag is affecting only the magnitudes of the specific angular momentum and eccentricity vectors, as it follows directly from Eqs. (11) by taking into account Eqs. (8). Also, the variational equation for the specific angular momentum is redundant, and therefore it will be omitted. Instead, the set of differential equations will be completed with the link between the new independent variable τ and the time t, as it follows from Eq. (22). The variational equations for the orbital elements, as well as for the time variable t, with respect to the new independent variable τ, are derived as:

$$\begin{cases} \bar{e}' = \dfrac{8\left[K\left(\bar{e}\right) - E\left(\bar{e}\right)\right]}{\pi\bar{a}\left(4 + 7\bar{e}^2\right)}; & \bar{e}\left(\tau_0\right) = \bar{e}_0 \\ \bar{a}' = \dfrac{8\left[2K\left(\bar{e}\right) - E\left(\bar{e}\right)\right]}{\pi\left(4 + 7\bar{e}^2\right)\left(1 - \bar{e}^2\right)}; & \bar{a}\left(\tau_0\right) = \bar{a}_0 \\ \bar{i}' = 0; & \bar{i}\left(\tau_0\right) = \bar{i}_0 \\ \overline{\Omega}' = \dfrac{2C_\Omega}{\sqrt{\mu}C_0\bar{a}^4\left(1 - \bar{e}^2\right)^3\left(4 + 7\bar{e}^2\right)}; & \overline{\Omega}\left(\tau_0\right) = \overline{\Omega}_0 \\ \overline{\omega}' = \dfrac{-2C_\omega}{\sqrt{\mu}C_0\bar{a}^4\left(1 - \bar{e}^2\right)^3\left(4 + 7\bar{e}^2\right)}; & \overline{\omega}\left(\tau_0\right) = \overline{\omega}_0 \\ \overline{M}' = \dfrac{-2}{C_0\bar{a}^2\left(4 + 7\bar{e}^2\right)\left(1 - \bar{e}^2\right)}\left(1 + \dfrac{C_M}{\sqrt{\mu}\bar{a}^2\left(1 - \bar{e}^2\right)^{3/2}}\right); & \overline{M}\left(\tau_0\right) = \overline{M}\left(t_0\right) \\ t' = \dfrac{-2}{C_0\sqrt{\mu\bar{a}}\left(4 + 7\bar{e}^2\right)\left(1 - \bar{e}^2\right)}; & t\left(\tau_0\right) = t_0 \end{cases} \tag{24}$$

Solution for small eccentricity: $e^4 \simeq 0$

The assumption of a relatively small eccentricity (i.e. $\bar{e}^4 \simeq 0$) is made at this point. By denoting:

$$y = \bar{e}^2 \tag{25}$$

and by expanding in Taylor series, Eqs. (24) are approximated as:

$$\begin{cases} y' = \dfrac{y}{\bar{a}}; \quad \bar{a}' = 1; \quad \bar{i}' = 0; \\ \overline{\Omega}' = \dfrac{C_\Omega}{2\sqrt{\mu}C_0\bar{a}^4} + \dfrac{5}{8}\dfrac{C_\Omega y}{\sqrt{\mu}C_0\bar{a}^4}; \\ \overline{\omega}' = -\dfrac{C_\omega}{2\sqrt{\mu}C_0\bar{a}^4} - \dfrac{5}{8}\dfrac{C_\omega y}{\sqrt{\mu}C_0\bar{a}^4}; \\ \overline{M}' = -\dfrac{1}{2C_0\bar{a}^2} - \dfrac{C_M}{2\sqrt{\mu}C_0\bar{a}^4} + \dfrac{3}{8}\left(\dfrac{1}{C_0\bar{a}^2} - \dfrac{C_M}{\sqrt{\mu}C_0\bar{a}^4}\right)y; \\ t' = \dfrac{3y - 4}{8C_0\sqrt{\mu\bar{a}}} \end{cases} \tag{26}$$

Note that the independent variable τ is exactly the averaged semimajor axis $\bar{a}$:

$$\bar{a} = \tau \tag{27}$$

and therefore the first equation is solved to obtain:

$$y = \frac{\bar{e}_0^2}{\bar{a}_0}\tau, \tag{28}$$

which offers the explicit expression of the averaged eccentricity with respect to the averaged semimajor axis:

$$\bar{e} = \frac{\bar{e}_0}{\sqrt{\bar{a}_0}}\sqrt{\bar{a}} \tag{29}$$

The rest of the equations transform into:

$$\begin{cases} \overline{\Omega}' = \dfrac{C_\Omega}{2\sqrt{\mu}C_0\tau^4} + \dfrac{5}{8}\dfrac{\bar{e}_0^2 C_\Omega}{\sqrt{\mu}\bar{a}_0 C_0 \tau^3} \\ \overline{\omega}' = -\dfrac{C_\omega}{2\sqrt{\mu}C_0\tau^4} - \dfrac{5}{8}\dfrac{\bar{e}_0^2 C_\omega}{\sqrt{\mu}\bar{a}_0 C_0 \tau^3}; \\ \overline{M}' = -\dfrac{1}{2C_0\tau^2} - \dfrac{C_M}{2\sqrt{\mu}C_0\tau^4} + \dfrac{3\bar{e}_0^2}{8\bar{a}_0}\left(\dfrac{1}{C_0\tau} - \dfrac{C_M}{\sqrt{\mu}C_0\tau^3}\right); \\ t' = \dfrac{3\bar{e}_0^2}{8C_0\bar{a}_0\sqrt{\mu}}\sqrt{\tau} - \dfrac{1}{2C_0\sqrt{\mu}}\dfrac{1}{\sqrt{\tau}}; \end{cases} ; \tag{30}$$

The explicit solution is obtained by taking into account the initial conditions written in Eqs. (24):

$$\begin{cases} \overline{\Omega} = \overline{\Omega}_0 + \left(\dfrac{-C_\Omega}{6\sqrt{\mu}C_0\tau^3} - \dfrac{5}{16}\dfrac{\bar{e}_0^2 C_\Omega}{\sqrt{\mu}\bar{a}_0 C_0\tau^2}\right)\Bigg|_{\tau=\bar{a}_0}^{\tau=\bar{a}}; \\ \overline{\omega} = \bar{\omega}_0 + \left(-\dfrac{C_\omega}{6\sqrt{\mu}C_0\tau^3} - \dfrac{5}{16}\dfrac{\bar{e}_0^2 C_\omega}{\sqrt{\mu}\bar{a}_0 C_0\tau^2}\right)\Bigg|_{\tau=\bar{a}_0}^{\tau=\bar{a}}; \\ \overline{M} = \overline{M}(t_0) + \left[\dfrac{1}{2C_0\tau} + \dfrac{C_M}{6\sqrt{\mu}C_0\tau^3} + \dfrac{3\bar{e}_0^2}{8\bar{a}_0}\left(\dfrac{1}{C_0}\ln\tau + \dfrac{C_M}{2\sqrt{\mu}C_0\tau^2}\right)\right]\Bigg|_{\tau=\bar{a}_0}^{\tau=\bar{a}}; \\ t = t_0 + \left(\dfrac{\bar{e}_0^2}{4C_0\bar{a}_0\sqrt{\mu}}\tau^{3/2} - \dfrac{1}{C_0\sqrt{\mu}}\sqrt{\tau}\right)\Bigg|_{\tau=\bar{a}_0}^{\tau=\bar{a}}, \end{cases} \tag{31}$$

where the constants C_Ω, C_ω and C_M are defined in Eq. (21). The explicit expression of the mean semimajor axis with respect to time $\bar{a} = \bar{a}(t)$ is determined from the last of Eqs. (31), by solving the cubic polynomial equation in $\zeta = \sqrt{\bar{a}/\bar{a}_0}$:

$$\zeta^3 - \frac{4}{\bar{e}_0^2}\zeta + \frac{4 - \bar{e}_0^2}{\bar{e}_0^2} - \frac{4(t - t_0)C_0\sqrt{\mu}}{\bar{e}_0^2\sqrt{\bar{a}_0}} = 0 \tag{32}$$

A discussion on the roots of the polynomial equation (32) does not make the subject of the present approach, since it involves only elementary algebraic manipulations. It may be proven that all the three solutions of the cubic equation (32) are real, in the conditions where the term not involving ζ is positive, and the solution $\zeta = \zeta(t)$ always satisfies: $0 < \zeta(t) \leq 1$.

Once the averaged semimajor axis $\bar{a} = \bar{a}_0 \zeta^2$ is obtained from Eq. (32), the averaged eccentricity $\bar{e}$ is obtained from Eq. (29), while the other three orbital elements are determined based on Eqs. (31).

In order to determine the averaged Cartesian position and velocity $(\bar{\mathbf{r}}, \bar{\mathbf{v}})$, only the classic Kepler equation needs to be solved to obtain the averaged eccentric anomaly $\bar{E}$ from the averaged mean anomaly $\bar{M}$.

Solution for very small eccentricity: $e^2 \simeq 0$

The equations above are susceptible of exhibiting numerical difficulties for very small eccentricities, due to the presence of the term $\bar{e}_0^2$ at the denominator of Eq. 32. In such situation, a different path should be followed in order to derive the explicit equations of motion, and the results will be significantly simpler.

Instead of the definition in Eq. (22), introduce the new independent variable $\tau = \tau(t)$ through the IVP:

$$\mathrm{d}t = \frac{-1}{2C_0\sqrt{\mu\bar{a}}}\mathrm{d}\tau, \;\; \tau(t_0) = \bar{a}_0 \tag{33}$$

The new variational equations with respect to τ become:

$$\begin{cases} \bar{e}' = \dfrac{2\left(1-e^2\right)\left[K(e) - E(e)\right]}{\pi a e}; & \bar{e}(\tau_0) = \bar{e}_0 \\ \bar{a}' = \dfrac{2\left[2K(\bar{e}) - E(\bar{e})\right]}{\pi}; & \bar{a}(\tau_0) = \bar{a}_0 \\ \bar{i}' = 0 & \bar{i}(\tau_0) = \bar{i}_0 \\ \overline{\Omega}' = \dfrac{C_\Omega}{2\sqrt{\mu}C_0\bar{a}^4\left(1-\bar{e}^2\right)^2} & \overline{\Omega}(\tau_0) = \overline{\Omega}_0 \\ \overline{\omega}' = \dfrac{-C_\omega}{2\sqrt{\mu}C_0\bar{a}^4\left(1-\bar{e}^2\right)^2}; & \overline{\omega}(\tau_0) = \overline{\omega}_0 \\ \overline{M}' = -\dfrac{1}{2C_0\bar{a}^2}\left[1 + \dfrac{C_M}{\bar{a}^2\sqrt{\mu}\left(1-\bar{e}^2\right)^{3/2}}\right] & \overline{M}(\tau_0) = \overline{M}(t_0) \\ t' = \dfrac{-1}{2C_0\sqrt{\mu\bar{a}}}; & t(\tau_0) = t_0 \end{cases} \tag{34}$$

By expanding Eqs. (34) in Taylor series with respect to e and by making the assumption $e^2 \simeq 0$, the following approximation is obtained:

$$\begin{cases} \bar{e}' = 0;\ \bar{a}' = 1;\ \bar{i}' = 0;\ t' = \dfrac{-1}{2C_0\sqrt{\mu\bar{a}}}; \overline{\Omega}' = \dfrac{C_\Omega}{2\sqrt{\mu}C_0\bar{a}^4}; \\ \overline{\omega}' = \dfrac{-C_\omega}{2\sqrt{\mu}C_0\bar{a}^4};\ \overline{M}' = -\dfrac{1}{2C_0\bar{a}^2}\left(1 + \dfrac{C_M}{\sqrt{\mu}\bar{a}^2}\right) \end{cases} \tag{35}$$

It follows that the eccentricity remains constant, while the semimajor axis is equal to the new independent variable, $\bar{a} = \tau$. The time variable t is derived as:

$$t' = \frac{-1}{2C_0\sqrt{\mu\tau}} \Rightarrow t = t_0 + \frac{\sqrt{\bar{a}_0} - \sqrt{\tau}}{C_0\sqrt{\mu}} \tag{36}$$

which leads to the explicit dependence of the averaged semimajor axis on time:

$$\bar{a}\left(t\right) = \left[\sqrt{\bar{a}_0} - \left(t - t_0\right)C_0\sqrt{\mu}\right]^2 \tag{37}$$

The expressions of $\bar{\Omega}$, $\bar{\omega}$ and M with respect to $\bar{a}$ are derived by integrating the rest of Eqs. (35). The time-explicit expressions for $\bar{\Omega}\left(t\right), \bar{\omega}\left(t\right)$ and $\bar{M}\left(t\right)$ are derived as follows:

$$\begin{cases} \overline{\Omega} = \overline{\Omega}_0 - \dfrac{C_\Omega}{6\sqrt{\mu}C_0}\left[\dfrac{1}{\bar{a}^3\left(t\right)} - \dfrac{1}{\bar{a}_0^3}\right] \\ \overline{\omega} = \bar{\omega}_0 + \dfrac{C_\omega}{6\sqrt{\mu}C_0\bar{a}_0^3}\left[\dfrac{1}{\bar{a}^3\left(t\right)} - \dfrac{1}{\bar{a}_0^3}\right] \\ \overline{M} = \overline{M}\left(t_0\right) + \dfrac{1}{2C_0}\left[\dfrac{1}{\bar{a}\left(t\right)} - \dfrac{1}{\bar{a}_0}\right] + \dfrac{C_M}{6C_0\sqrt{\mu}}\left[\dfrac{1}{\bar{a}^3\left(t\right)} - \dfrac{1}{\bar{a}_0^3}\right] \end{cases} \tag{38}$$

Like in the previous case, the Cartesian state $\left(\bar{\mathbf{r}}, \bar{\mathbf{v}}\right)$ is obtained by solving the Kepler equation to obtain the averaged eccentric anomaly $\bar{E} = \bar{E}\left(\bar{M}\right)$.

CONCLUSIONS

The paper introduces a straightforward averaging method in order to determine the equations of motion of a satellite orbiting Earth under the influence of the J_2 zonal harmonic and the atmospheric drag. The equations for the classical averaged orbital elements are obtained in a time-explicit form. Two versions of the same model are determined, namely for small ($e^4 \simeq 0$) and very small ($e^2 \simeq 0$) eccentricities. The obtained equations of motion prove to be not more complicated than their J_2-only counterparts.

The present method will be used in order to develop an analytic short-term propagator for low orbiting Earth satellites. Further developments will assume more complicated models of the atmosphere, namely an exponential distribution of its density, as well as an elliptic shape.

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