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OPTIMAL CONTROL ON GAUSS' EQUATIONS

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This paper presents some theoretical developments on optimal control applied to Gauss' equations, when only control perturbations are considered. For a large family of problems, it is shown that the optimal control points toward a primer vector (equivalent to Lawden's primer vector in cartesian coordinates). The primer vector is a compromise between feedback laws (named here *control primitives*), with the costates acting as time-varying weights. The general state and costate equations are provided together with the linearized equations needed for the propagation of the partials. The first and second derivatives of Gauss equations in Keplerian and equinoctial elements are presented in a compact form, which is easy to derive and to code using cartesian tensor notation and logarithmic differentiation. Finally it is analytically proven that the optimal control maximizing the final semi-major axis is not always tangent; although it is it confirmed that a constant, tangential thrust (which is the control primitive associated to the semi-major axis) represent a good approximation.

INTRODUCTION

Over the last decade low-thrust propulsion has been implemented more and more frequently to deep space missions[1,2] and Earth-orbiting satellites,[3] thanks to the mass savings resulting from its high specific-impulse engines. Yet the design of low-thrust trajectories is still a challenging task, and new techniques are developed every year to this purpose.

The indirect-method approach searches for candidate solutions that satisfy the necessary conditions from optimal control theory.[4,5] The control is expressed as a function of the states and of an additional set of variables, the costates, which are integrated along the optimal trajectory. The necessary conditions include the equation of motions for the costates, and a set of mixed boundary conditions (transversality conditions), which transform the optimal control problem into a two-point, boundary-value problem.

Typically, indirect methods model the spacecraft dynamics with two-body problem equations and control accelerations in Cartesian coordinates.[6] With this choice of coordinates, the control direction is parallel to the primer vector,[7,8,9] which is the costate associated to the velocity.

More rarely has the indirect method been applied to Gauss' equations, which provide the rate of change of a set of orbital elements due to external accelerations.[10] Some common choices for the orbital elements are Keplerian elements and equinoctial (nonsingular) elements.[11] In the 90's Kechichian published a series of papers[12,13,14,15,16] applying optimal control theory to Gauss' equation with equinoctial elements for various choices of the fast variable. Kechichian derived the optimal control and the costate equation, and provided the partial derivatives of the matrix of coefficients, as these derivatives appear in the costate equations. Since then, other scholars have applied the equinoctial element formulation to low-thrust optimization tools,[17] with the inclusion of perturbing forces other than the control accelerations.[18] Also generic formulas are found for the minimum-time or minimum-propellant mass problems[19,20] . Other works[21] applied optimal control theory to the averaged equations.

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Alternative common approaches to compute low-thrust trajectory with Gauss equations provide feedback laws,[22,23,24,25] which depend on some parameters that can be optimized.[26] Several special solutions were computed for a specific control law (see a discussion on this in Petropoulos[27]). Other solutions are obtained manipulating the equations of motion and introducing some approximations[28] - for example replacing the time variable with the fast variable, using averaging,[29] or using first order expansions on the control magnitude.[30,31]

In this paper, the necessary conditions from optimal control theory are applied directly to Gauss's equation, at first for a generic set of orbital elements and then using Keplerian and equinoctial elements. A first objective of this work is to provide new insight, rather than a new tool, on low-thrust trajectory optimization. Feedback laws called here *optimal control primitives* are found to play an important role in the definition of the control direction. The control direction is found to follow a primer vector (equivalent to Lawden's[7] in Cartesian coordinates) for a large class of problems, even when additional perturbations are included. In an example application, the tangential thrust is analytically proven not to satisfy the necessary conditions for semi-major axis maximization problem - a result that was hinted at in the past.[30,32,33]

A second goal of this paper is to provide the full set of equations to solve the two-point boundary value problem with Keplerian and equinoctial elements. Besides from the state and costate equations, the equations of motion of the linearized system are also derived, and used to compute the partial derivatives of the final states and costates with respect to the initial states and costates. While the state and costate equations for Equinoctial elements can already be found in literature,[13,17] the same equations for Keplerian elements have not been published yet to our knowledge. Also, the derivatives of such equations needed in the linearized system (i.e. the second derivatives of the coefficient in Gauss equations) are published here for the first time, for both equinoctial and Keplerian elements. In the past, the derivation and coding of these derivatives have been tedious and error-prone, and for this reason, other means of differentiation have been used (such as complex derivatives,[34,35] or automatic differentiation), accepting some penalty in the computation time. In this paper, all the derivatives are derived using Cartesian tensor notation and logarithmic differentiation. The method used results in compact formulas, which are easily coded using nested loops. Alternatively, symbolic manipulation software such as Maple or Mathematica could be used, although in some cases this could result in longer expressions.

PRELIMINARY MATERIAL

Tensor index notation

Throughout this work the Cartesian tensor notations is used, where an N-dimensional array X is represented by its elements $X_{i1,...,iN}$. When an equation contains repeated indices, a summation is implied over their range (Einstein summation convention). The *kroneker* δ_{ij} (1 for $i = j$, 0 otherwise) is used to isolate specific components of the array.

The Cartesian tensor notation is a powerful tool to derive and write lengthy derivatives in a simple and compact form. Higher order derivatives are computed applying the chain rule repetitively. Table 1 shows an example scalar function and an example vector function and their derivatives, expressed on the left with conventional traditional vector notation (with bold font for column vectors), and on the right in components with Cartesian tensor notation.

Gauss' equations

The restricted two-body problem perturbed by a generic acceleration u_i is represented in Cartesian coordinates with the equations

$$\begin{cases} \dot{r}_i &= v_i \\ \dot{v}_i &= -\frac{\mu}{|r|^3} r_i + u_i \end{cases} \qquad (1)$$

vector	components
$\varphi(\mathbf{x}) = \|\mathbf{x}\|$	$\varphi(x) = \|x\| = \sqrt{x_k x_k}$
$\frac{d\varphi(\mathbf{x})}{d\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$	$\left(\frac{d\varphi(x)}{dx}\right)_i = \frac{d\varphi(x)}{dx_i} = \frac{x_k \delta_{ki}}{\sqrt{x_k x_k}} = \frac{x_i}{\|x\|}$
$\frac{d\varphi(\mathbf{x}(t))}{dt} = \left(\frac{d\varphi(\mathbf{x})}{d\mathbf{x}}\right)^T \frac{d\mathbf{x}}{dt} = \frac{\mathbf{x}^T \dot{\mathbf{x}}}{\|\mathbf{x}\|}$	$\frac{d\varphi(x(t))}{dt} = \frac{d\varphi(x)}{dx_i}\frac{dx_i}{dt} = \frac{x_i \dot{x}_i(t)}{\|x\|}$
$\mathbf{f}(\mathbf{x}) = \frac{g(\mathbf{x})}{\|g(\mathbf{x})\|^n}$	$f_i = \frac{g_i}{\|g(x)\|^n}$
$\frac{d\mathbf{f}(\mathbf{x})}{d\mathbf{x}} = \frac{1}{\|f\|^n}\frac{d\mathbf{g}}{d\mathbf{x}} - n\frac{\mathbf{g}}{\|f\|^{n+2}}(\mathbf{g}^T \frac{d\mathbf{g}}{d\mathbf{x}})$	$\left(\frac{df}{dx}\right)_{ij} = \frac{df_i}{dx_j} = \cdots = \frac{1}{\|g\|^n}\frac{dg_i}{dx_j} - n\frac{g_i g_k}{\|g\|^{n+2}}\frac{dg_k}{dx_j}$
$\frac{d\mathbf{f}(\mathbf{x}(t))}{dt} = \left(\frac{\dot{\mathbf{g}}}{\|\mathbf{g}\|^n} - n\frac{\mathbf{g}(\mathbf{g}^T\dot{\mathbf{g}})}{\|f\|^{n+2}}\right)$	$\left(\frac{df}{dt}\right)_i = \frac{df_i}{dx_j}\frac{dx_j}{dt} = \cdots = \frac{\dot{g}_i}{\|g\|^n} - n\frac{g_i g_k \dot{g}_k}{\|g\|^{n+2}}$

Table 1. Cartesian tensor notation

If the state of the spacecraft is provided by a set of elements α (such as the Keplerian or the equinoctial elements), Eq. 1 become Gauss' equations, which take the general from:

$$\dot{\alpha}_i(t) = F_{ij}(\alpha(t))u_i + c_i(\alpha(t)) \tag{2}$$

This section recalls some relevant properties of Gauss' equations. First a quick derivation of Eq. 2 (similar to that in[36]) is presented. Then the expressions for $F(\alpha)$ and $c(\alpha)$ are reviewed for some common choice of α (their derivation can be found in the literature[10,14]).

Derivation of the general form of Gauss' Equation When a dynamical system $\dot{x} = f(x,t) \quad x \in \mathbb{R}^n$ is integrable, it possesses n constants of motion $\varphi_i(x,t), i = 1, \ldots, n$ with zero Lie derivative (directional derivative) along the flow:

$$\pounds_f \varphi_i = \frac{d}{dt}\varphi_i\left(x(t),t\right) = \left(\frac{\partial \varphi_i}{\partial x_j} f_j + \frac{\partial \varphi_i}{\partial t}\right)_{x(t),t} = 0 \tag{3}$$

If external forces $g(x,u,t)$ are added to the dynamical system, the functions φ_i are not constant anymore. Using Eq. 3, their rates of change become

$$\frac{d}{dt}\varphi_i\left(x(t),t\right) = \left(\frac{\partial \varphi_i}{\partial x_j} g_j\right)_{x(t),t} \tag{4}$$

An application of Eq. 4 to the system of Eq. 1 yields to

$$\dot{\alpha}_i(t) = F_{ij}(\alpha(t),t)u_i \tag{5}$$

where α is a set of constants of motion defined by the transformations

$$\alpha = \varphi(r,v,t), \qquad r = r(\alpha,t), \qquad v = v(\alpha,t) \tag{6}$$

and

$$F_{ij}(\alpha,t) = \left(\frac{d\varphi_i}{dv_j}\right)_{r(\alpha,t),v(\alpha,t)} \tag{7}$$

Gauss's equations take the form of system Eq. 5 when α is a set of constants of motions; the system is linear in the accelerations, and non-autonomous, because the derivative of the time of the pericenter passage includes a term linear in t.[10] However, if the constant of motion associated to the time of the pericenter passage is replaced by a fast-moving variable such as true anomaly f or true longitude L, then the coordinate transformation Eq. 6 becomes time independent, and Gauss' equations take the form of the system Eq. 2, which is autonomous and affine in the acceleration , with

$$F_{ij}(\alpha) = \left(\frac{d\varphi}{dv}\right)_{r(\alpha),v(\alpha)} \tag{8}$$

$$c_i(g) = \left(\frac{d\varphi_i}{dr_j}v_j - \frac{\mu}{r^3}\frac{d\varphi_i}{dv_j}r_j\right)_{r(\alpha),v(\alpha)}$$

The additional terms c_i appear for the equations associated to the fast-moving variables. The fact that the system is autonomous and affine in the control is relevant for the discussion on the optimal control in the next section

In Eq. 2 the accelerations are expressed in an inertial reference frame. When the accelerations are expressed in a local reference frame (such as polar coordinates RSH or tangential coordinates TNH), then the components of the velocities in Eq. 7 are considered in the same frame, i.e.:

$$\dot{\alpha} = F(\alpha)u + c(\alpha) = F^{loc}(\alpha)u^{loc} + c(\alpha)$$

where $u^{loc} = A(\alpha)u$, $v^{loc} = A(\alpha)v$, and

$$F^{loc}(\alpha) = \left(\frac{d\varphi}{d\tilde{v}}\right)_{r,\tilde{v}} = \left(\frac{d\varphi}{dv}\right)_{r,v} A^{-1}(\alpha) = F(\alpha)A^{-1}(\alpha)$$

In particular, to map the matrix $F(\alpha)$ from two different local frame, we use

$$F^{loc1}(\alpha) = F^{loc2}(\alpha)A^{loc1\to loc2}(\alpha) \tag{9}$$

Gauss' Equations with Keplerian Elements If α is the set of Keplerian orbital elements $\alpha = (a, e, i, \Omega, \theta, f)$, where θ is the argument of latitude $\theta = f + \omega$, and the acceleration components are in the TNH frame, the matrix $F(\alpha)$ and the vector $c(\alpha)$ are[10]

$$F^{TNH}(\alpha) = \begin{bmatrix} \frac{2a^2v}{\mu} & 0 & 0 \\ \frac{2(e+\cos f)}{v} & -\frac{r\sin f}{av} & 0 \\ 0 & 0 & \frac{r\cos\theta}{h} \\ 0 & 0 & \frac{r\sin\theta}{h\sin i} \\ 0 & 0 & -\frac{r\sin\theta\cos i}{h\sin i} \\ -\frac{2\sin f}{ev} & -\frac{2}{v} - \frac{r\cos f}{vea} & 0 \end{bmatrix} \tag{10}$$

$$c_5(\alpha) = c_6(\alpha) = \frac{h}{r^2} \tag{11}$$

with the common formulas for momentum, radius and velocity:

$$h = \sqrt{\mu a\left(1-e^2\right)} \quad r = \frac{a\left(1-e^2\right)}{1+e\cos f} \quad v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

The choice of $\dot{\theta}$ in place of $\dot{\omega}$ as fifth Keplerian element simplifies the analysis in the following section, since $\dot{\theta}$ only depends on u_3. If needed, the expressions for the equation $\dot{\omega}$ and its derivatives are found summing those for $-f$ and θ. Finally, if the acceleration is defined in the polar frame, the matrix F^{RSH} is computed with Eq. 9 and with

$$A^{RSH\to TNH}(\alpha) = \frac{\mu}{vh}\begin{bmatrix} e\sin f & 1+e\cos f & 0 \\ -1-e\cos f & e\sin f & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Gauss' Equations with Equinoctial Elements The equinoctial elements[11] $\alpha = (a, P_1, P_2, Q_1, Q_2, L)$ are used to remove the singularities in Eq. 10 at zero eccentricity and inclination. They are related to the orbital elements with the formulas[10] $P_1 = e\sin(\Omega+\omega)$, $P_2 = e\cos(\Omega+\omega)$, $Q_1 = \tan\frac{i}{2}\sin\Omega$, $Q_2 = \tan\frac{i}{2}\cos\Omega$, while L is the true longitude $L = \Omega+\omega+f$. Using the true longitude as fast variable does not require solving Kepler's equation, and was proposed by Kechichian[14] . When the accelerations are in polar frame:

$$F^{RSH}(\alpha) = \begin{bmatrix} \frac{2a^2}{h}(P_2\sin L - P_1\cos L) & \frac{2a^2}{h}A & 0 \\ -AB\cos L & B\left(P_1 + (1+A)\sin L\right) & BP_2\left(Q_2\sin L - Q_1\cos L\right) \\ AB\sin L & B\left(P_2 + (1+A)\cos L\right) & -BP_1\left(Q_2\sin L - Q_1\cos L\right) \\ 0 & 0 & \frac{B}{2}\left(1+Q_1^2+Q_2^2\right)\sin L \\ 0 & 0 & \frac{B}{2}\left(1+Q_1^2+Q_2^2\right)\cos L \\ 0 & 0 & B(P_1\sin L - P_2\cos L) \end{bmatrix}$$

$$c_6 = (Bh)^{-1}$$

where

$$h = \sqrt{\mu a\left(1-P_1^2-P_2^2\right)} \qquad A = 1 + P_1\sin L + P_2\cos L \qquad B = \frac{h}{\mu A}$$

Note that A is $p/r = 1 + e\cos f$. If the accelerations are expressed in the tangential-normal frame, F^{TNH} is computed with Eq. 9 and

$$A^{TNH\rightarrow RST}(\alpha) = \frac{1}{\sqrt{A^2 + (P_2\sin L - P_1\cos L)^2}} \begin{bmatrix} P_2\sin L - P_1\cos L & A & 0 \\ -A & P_2\sin L - P_1\cos L & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

where we used the formulas for the radial and transverse velocity $v_R = \frac{A}{B}\left(P_2\sin L - P_1\cos L\right)$ and $v_S = \frac{1}{B}$.

Optimal control problems

Optimal control problems seek the minimum of a functional

$$J(u, x, t_I, t_F) = \varphi\left(t_I, t_F, x(t_I), x(t_F)\right) + \int_{tI}^{tF} L\left(x(t), u(t), t\right) dt \tag{14}$$

subject to the dynamical system

$$\dot{x}(t) = f(x(t), u(t), t) \tag{15}$$

with constraints on the boundary conditions

$$\psi(t_0, t_f, x(t_0), x(t_f)) = 0 \tag{16}$$

Here $x(t) \in \mathbb{R}^n$, $\psi(t_I, t_F, x_I, x_F) \in \mathbb{R}^p$, $p \leq n+2$, $\varphi(t_I, t_F, x_I, x_F) \in \mathbb{R}$, $u(t) \in U$ with $U \subseteq \mathbb{R}^m$. The functions f, $\frac{df}{dx}$, L, $\frac{dL}{dx}$ are assumed continuous; φ and ψ are differentiable; and u is piecewise continuous. At this point, it is convenient to introduce the costate functions $\lambda : [t_I^*, t_f^*] \rightarrow \mathbb{R}^n$ and the Hamiltonian function:

$$H(x, u, \lambda, t) = \lambda_i f_i(x, u, t) + L(x, u, t)$$

In optimal control theory, the search for the minimum becomes the search for solutions that satisfy the necessary conditions for optimality. In general, these conditions are not sufficient, therefore the solution found might be just a local minimum, and in fact, does not even need to be a minimum. Nevertheless, the necessary conditions for optimality are a powerful tool to select candidate solutions. This work implements the necessary conditions provided by the Pontryagin principle:[4]

Pontryagin Minimum Principle (PMP) *Let $u^*(t)$ be the optimal control that minimizes the functional Eq. 14, and let $x^*(t)$ be the corresponding optimal trajectory, between the optimal times t_I^*, t_F^*. Then there exists a function $\lambda^* : [t_I^*, t_f^*] \to \mathbb{R}^n$ and a constant $\lambda_0^* \geq 0$, and a such that*

1. $(\lambda^*(t), \lambda_0^*) \neq (0, 0)$ for any $t \in [t_I^*, t_F^*]$
2. *$x^*(t)$ and $\lambda^*(t)$ satisfy the equations*

$$\begin{aligned} \dot{x}^* &= \tfrac{dH}{d\lambda}(x^*, u^*, \lambda^*, \lambda_0^*, t) \\ \dot{\lambda}^* &= -\tfrac{dH}{dx}(x^*, u^*, \lambda^*, \lambda_0^*, t) \end{aligned} \tag{17}$$

with boundary conditions Eq. 16 and with additional boundary conditions ("transversality conditions")

$$H|_{t_I*} = \frac{d\Phi}{dt_I}, \qquad \lambda^*(t_I) = -\frac{d\Phi}{dx_I} \tag{18}$$

$$H|_{t_F*} = -\frac{d\Phi}{dt_F}, \qquad \lambda^*(t_F) = \frac{d\Phi}{dx_F} \tag{19}$$

where the scalar functions H ("Hamiltonian") and Φ are defined as

$$H(x, u, \lambda, \lambda_0, t) = \lambda_i f_i(x, u, t) + \lambda_0 L(x, u, t)$$

$$\Phi(t_I, t_F, x_I, x_F \nu) = \varphi(t_I, t_F, x_I, x_F) + \nu_i \psi_i(t_I, t_F, x_I, x_F)$$

and $\nu \in \mathbb{R}^p$.

3. *For each $t \in [t_I^*, t_f^*]$, the function $u \to H(x^*(t), u, \lambda^*(t), \lambda_0^*, t)$ has a global minimum at $u = u^*(t)$, i.e., the optimal control u^* satisfies*

$$u^*(t) = \arg\min_{u \in U} H\left(x^*(t), u, \lambda^*(t), \lambda_0^*\right) \tag{20}$$

for any $t \in [t_0, t_f]$ and for any $u \in U$.

4. *If the Hamiltonian is autonomous, than is it constant along the optimal trajectory. More in general*

$$\frac{d}{dt} H(x^*(t), u^*(t), \lambda^*(t), \lambda_0^*, t) = \frac{\partial}{\partial t} H(x^*(t), u^*(t), \lambda^*(t), \lambda_0^*, t)$$

The Pontryagin necessary conditions include a multiplier λ_0 in front of the Lagrangian, and the condition that $(\lambda(t), \lambda_0)$ never vanish. λ_0 is called the abnormal multiplier, and in general it is assumed $\neq 0$, and further normalized to 1. When no solutions can be found for $\lambda_0 = 1$, then one should check the *abnormal* case $\lambda_0 = 0$ (with $\lambda(t) \neq 0$). In this paper, however, optimal control exists with $\lambda_0 = 1$.

The Pontryagin Principle is typically stated with some variation for different type of problems , such as fixed final-time, fixed final-state, etc, hence the name of *principle*, rather than just *theorem*. This section presents a more compact notation for the transverality conditions Eq. 18-19, which are valid for all boundary constraints of the type in Eq. 16 . In particular it is easy to check that: (1) if the initial time and states are fixed, then $n + 1$ elements of ψ are $t_I - a = 0, x_I - b = 0$, which yields to H_I and λ_I being free (i.e. equal to undefined free multipliers ν_i); (2) if t_f is free, then $H_F = 0$ works as a stopping condition for the forward propagation of the equation of motion, and if the Hamiltonian does not depend on time, then $H\left(x^*(t), u^*(t), \lambda^*(t), t\right) = 0$; (3) if $x_i(t_F)$ is neither constrained, nor included in φ, then the corresponding final costate vanishes at final time: $\lambda_i(t_F) = \frac{d\Phi}{dx_{F,i}} 0$; (4) if $x_i(t_F)$ is fixed, then one element of ψ is $x_{F,i} - a = 0$ and the final costate $\lambda_i(t_F)$ is free (i.e. equal to the undefined corresponding multipliers).

Note on the necessary conditions and on the interpretation of the costates Besides from Pontryagin Principle, two other approaches can be used to define necessary conditions similar to Eq. 17-20: calculus of variations and dynamic programming. The three approaches are briefly reviewed as they provide different insight on the different the costate functions. For a more detailed treatment on these topics, the interested reader is referred to the relevant literature.

The first approach to derive the necessary conditions traces back to the works of Euler, Bernoulli, and Lagrange on calculus of variations.[37] In calculus of variations, parametric families of curves are defined around a nominal trajectory: for instance, the family of curves $u(\epsilon, t)$ is defined such as $u(0,t) = u^*(t)$ is the optimal control. Weak variations of $u^*(t)$ are then defined as $\delta u(t) = \frac{d}{d\epsilon} u(\epsilon, t)\big|_{\epsilon=0}$, while the first variation of the functional $J(u)$ is computed as $\delta J = \frac{d}{d\epsilon} J(u(\epsilon, t))\big|_{\epsilon=0}$. The necessary condition for $u^*(t)$ to minimize $J(u)$ is then simply $\delta J(u) = 0$. However, when applying calculus of variation to optimal control problems, the minimization of $J(u)$ is constrained by the dynamical system Eq. 15. The necessary conditions for the constrained problem are found adding the dynamic constrains inside the integral Eq. 14 through the Lagrange multipliers (see parameter optimization with constraints). The costate function $\lambda_i(t)$ is then the Lagrange multiplier associated to the equation for $\dot{x}_i$ in Eq.15.The calculus-of-variation approach typically requires some regularity conditions, such as $u(t)$ being differentiable and with an open range; f, L being differentiable in u. Also, calculus of variation yields to a weak condition on H, which is only locally minimized by u *. Fore more details on the calculus of variation approach, see for example Bryson and Ho.[5]

The Minimum Principle was formulated by Portraying and his students in 1956.[4] The principle applies also to piecewise-continuos controls, and yields to the stronger condition Eq. 20: at any point of the optimal trajectory, the optimal control $u^*(t)$ is the *global* minimizer of the Hamiltonian over the set of allowed controls U. The global minimization of H should not be confused with the minimization of the functional J(u). Even if H is globally minimized, the optimal control and the optimal trajectory found are only a candidate of a local minimum for the original optimal control problem. The proof of the PMP offers another interpretation of the costate functions: the final costate $\lambda(t_F)$ is the normal to a hyperplane separating the terminal cone (tangent space to the set of attainable terminal conditions), from a cost-reduction direction. The costates are defined in the dual space of the optimal trajectory, and their coupling (dot product) with tangent vectors is constant along the flow - a property which ultimately requires the Hamiltonian be globally minimized along the trajectory. It is clear from the proof that in general the final hyperplane, and hence the final costate, is not unique. A concise proof can be found in Liberzon.[38]

In both the Pontryagin principle and in calculus of variation, the minimization of H is used to find an expression of u^* as function of x^* and λ^*. Since the optimal costates are only found by satisfying the transversality conditions, the optimal control problem is solved open-loop. Instead, the dynamic programming necessary conditions provide a closed loop solution for the optimal control; the conditions are also sufficient for a global minimum. Dynamic programming was developed by Bellman around the same time that Pontryagin was working on the Maximum Principle. Bellman first defines an optimal cost-to-go function $V(x, t)$, which depends on the initial state and time only. Applying the principle of optimality to the cost-to-go function yields to the Hamilton-Jacobi-Bellman equation, which is a partial differential equation that, if solved, provides an expression for $V(x, t)$ and for the optimal control. Unfortunately solving HJB is hard even for simple problems. In the dynamic programming formulation, the costate functions are the sensitivity of the cost-to-go function to the initial conditions, i.e. $\lambda(x, t) = dV/dx$.

OPTIMAL CONTROL ON GAUSS' EQUATIONS

This section applies the necessary conditions of Pontryagin's principle to Gauss' equations. Here the functional is

$$J(u, \alpha, t_I, t_F) = \varphi\left(t_I, t_F, \alpha(t_I), \alpha(t_F)\right) + \int_{tI}^{tF} L\left(u(t)\right) dt \tag{21}$$

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*(for example dH/du=0, d2H/du>=0, where H can include multipliers for bounded controls)

and the dynamical system is

$$\dot{\alpha}_i(t) = f(\alpha, u) = F_{ij}(\alpha(t))u_i + c_i(\alpha) \tag{22}$$

with $u(t) \in U$. The control Hamiltonian is the autonomous function

$$H(\alpha, u, \lambda) = L(u) + \lambda_i \left(F_{ij}(\alpha)u_j + c_i(\alpha)\right) = L(u) + p_j(\alpha, \lambda)u_j + \lambda_i c_i(\alpha) \tag{23}$$

where we introduced the function

$$p_j(\alpha, \lambda) = \lambda_i F_{ij}(\alpha) \tag{24}$$

The costate equation is

$$\dot{\lambda}_i^* = -\frac{dH}{dx_i} = -\lambda_k^* \left(\frac{dF_{kj}}{d\alpha_i}u_j^* + \frac{dc_k}{d\alpha_i}\right) \tag{25}$$

which implies $\lambda^*(t) \neq 0$ for every t (if $\lambda^*(t) = 0$ for any t, then $\lambda^*(t) = 0$ for all the t, which would violate the transversality conditions).

The optimal control is found by minimizing H over the set of admissible controls, and its expression as function of states and costates depends on L and on U. Table 2 shows a list of common optimal control problems. In all cases, the control direction is provided by the vector p. In some cases, the following lemmas are used, which concern the constrained minimization of the scalar affine function

$$y(x) = a + b_i x_i \tag{26}$$

where $a, y(x) \in \mathbb{R}$, and $b, x \in X \subseteq \mathbb{R}^m$

Lemma(1) Assume X is the annulus defined by $0 \leq x_L \leq |x| \leq x_U$. If $b_i \neq 0$ for at least one i, then $y(x)$ has a unique, global minimum at $x^* = -x_U p/|p|$.

Lemma(2) Assume X is the hypercube defined by $0 \leq x_i \leq x_{U,i}$. If $b_i \neq 0$ for every i , then $y(x)$ has a unique, global minimum defined by

$$\begin{cases} x_i^* = & 0 & \text{if } b_i > 0 \\ x_i^* = & x_{U,i} & \text{if } b_i < 0 \\ x_i^* = & x \in [0, x_{U,i}] & \text{if } b_i = 0 \end{cases}$$

For minimum control effort problems (case I) , H is quadratic in $u \in \mathbb{R}^m$, and the optimal control is opposite to p. In the other problems, the Hamiltonian is affine in either u or $||u||$, and therefore an optimum exists only if u is bounded. In practical applications, the control magnitude is limited by a maximum value, which, without loss of generality, is assumed to be 1 (this is always possible with the proper non-dimensionalization). The resulting optimal controls are shown in Table 2, and are derived using Lemma1 and Lemma 2 on either u or $||u||$ and $u/||u||$. Case (II) is the minimum Δv problem, which includes coast arcs if $||p|| < 1$. The last row shows a generic Meyer problem, i.e. with $L = 0$ and $\varphi \neq 0$. The optimal control is the same as the for case (III), minimizing the final time. The last column of Table 2 shows $H^*(\alpha^*, \lambda^*)$, which is the Hamiltonian evaluated along the optimal trajectory. Since Eq. 23 is autonomous, the Hamiltonian is constant.

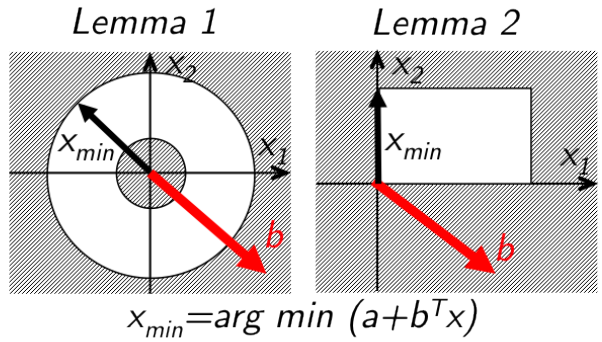

Figure 1. Lemma 1 and 2

Problem	U	$L(u)$	$u^*(p)$	$H^*(\alpha^*, \lambda^*)$
(I) Control effort	$\mathbb{R}^n$	$\frac{1}{2}\|u\|^2$	$-p$	$-\frac{\|p(\alpha^*,\lambda^*)\|^2}{2} + \lambda_i^* c_i(\alpha^*)$
(II) Δv	$B_1(0)$	$\|u\|$	$\begin{cases} -\frac{p}{\|p\|} & \text{if} \quad \|p\| > 1 \\ 0 & \text{if} \quad \|p\| < 1 \end{cases}$	$\begin{cases} 1 - \|p(\alpha^*, \lambda^*)\| + \lambda_i^* c_i(\alpha^*) & \text{if} \quad \|p\| > 1 \\ \lambda_i^* c_i(\alpha^*) & \text{if} \quad \|p\| < 1 \end{cases}$
(III) Time of flight	$B_1(0)$	1	$-\frac{p}{\|p\|}$	$1 - \|p(\alpha^*, \lambda^*)\| + \lambda_i^* c_i(\alpha^*)$
(IV) Meyer	$B_1(0)$	0	$-\frac{p}{\|p\|}$	$\|p(\alpha^*, \lambda^*)\| + \lambda_i^* c_i(\alpha^*)$

Table 2. Optimal control problems on Gauss' equations

Primer vector and optimal control primitives

In all cases of Table 2, the vector p determines the control much like the Lawden's primer vector[7,8] determines the control when the equations are in Cartesian coordinates. For this reason, p will be referred to as the primer vector. We also defines the *Optimal Control Primitives* as the functions $\pi^{(i)}$

$$\pi_j^{(i)}(\alpha) = F_{ij}(\alpha)$$

that is, the control primitive $\pi^{(i)}(\alpha)$ is the i-th row of $F(\alpha)$. The optimal control primitives have a number of interesting properties, in particular:

1. $\pi^{(i)}(\alpha)$ is the gradient of α_i with respect to the velocity (see eq. 7)
2. equivalently, $\pi^{(i)}$ is the feedback law that maximizes the rate of change of the element α_i This can be also verified using Lemma 1:

$$\arg\max_{\|u\| \leq 1} \dot{\alpha}_i = \arg\max_{\|u\| \leq 1} (F_{ij} u_j + c_i) = \arg\max_{\|u\| \leq 1} \left(\pi_j^{(i)} u_j + c_i\right) = \frac{\pi^{(i)}(\alpha)}{||\pi^{(i)}(\alpha)||}$$

3. The primer vector is a linear combination of $\pi^{(i)}$, with time-varying coefficients:

$$p_j(\alpha, \lambda) = \lambda_i F_{ij}(\alpha) = \sum_i \lambda_i \pi_j^{(i)}(\alpha) \tag{27}$$

Eq. 27 offers an interesting interpretation of the primer vector and of the costates: the optimal control direction is a compromise between locally maximizing control directions (given by the primitives), with the costates acting as time-varying weights. Of course the knowledge of the costates is still needed to find the primer vector; like when the spacecraft trajectory is formulated in Cartesian coordinates. In that case, however, the primer vector is simply the costate of the velocity, and the primitives would be zero or unit vectors. With a proper choice of α, however, one could hope that some of the information on the optimal control is already included in the functions $\pi^{(i)}$. A follow-up paper will investigate the use of $\pi^{(i)}$ to generate feedback laws, and compare this formulation with existing approaches like the Q-LAW[25] or MANTRA.[26]

Example application: Keplerian orbital elements and tangent-normal components

This section presents Gauss' equations with accelerations in the TNZ frame and orbital orbital elements $\alpha = (a, e, i, \Omega, \theta, f)$. The matrix $F(\alpha)$ and the vector $c(\alpha)$ are in Eq. 10-11 . Since many elements of F and c are zero, the primer vector and the costate equations become

$$p_m = \lambda_j F_{jm} \rightarrow \begin{cases} p_1 = F_{11}\lambda_1 + F_{21}\lambda_2 + F_{61}\lambda_6 \\ p_2 = F_{22}\lambda_2 + F_{62}\lambda_6 \\ p_3 = F_{33}\lambda_3 + F_{43}\lambda_4 + F_{53}\lambda_5 \end{cases} \tag{28}$$

$$\dot{\lambda}_i = -\frac{d(h/r^2)}{d\alpha_i}(\lambda_5 + \lambda_6) - \lambda_k \frac{dF_{kj}}{d\alpha_i} u_j^* \tag{29}$$

where the expressions for $\frac{d(h/r^2)}{d\alpha}$ and $\frac{dF_{ij}}{d\alpha_k}$ are found in the next section of the paper. We first note that the equation on λ_4 is $\dot{\lambda}_4 = 0 \rightarrow \lambda_4 = const$, since no components of $F(\alpha)$ depends on Ω; if moreover either the initial or the final Ω is free, then the transversality conditions yields to $\lambda_4(t_F) = \lambda_4 = 0$.

Since the first, second, and last row of F and c do not depend on the elements (i, Ω, θ), the corresponding costate equations on $(\lambda_3, \lambda_4, \lambda_5)$ can be written in the compact form

$$\dot{\lambda}_i = -\left(\lambda_3 \frac{dF_{33}}{d\alpha_i} + \lambda_4 \frac{dF_{43}}{d\alpha_i} + \lambda_5 \frac{dF_{53}}{d\alpha_i}\right) u_3^* \qquad i = 3, 4, 5 \tag{30}$$

Eq. 30 shows that if $(\lambda_3, \lambda_4, \lambda_5)$ vanish at any time, they vanish at all time, and from Eq.28, also p_3 would vanish, making the motion planar (as expected). As an illustrative example, we consider the maximization of the final semi-major axis $\varphi = -a_F$, with fixed initial conditions and final time, and free final states $\alpha_i(t_F)$. The boundary conditions are:

$$\psi(\alpha(t_I), t_I, t_F) = \begin{pmatrix} a(t_I) - \tilde{a}_I \\ e(t_I) - \tilde{e}_I \\ i(t_I) - \tilde{i}_I \\ \Omega(t_I) - \tilde{\Omega}_I \\ \theta(t_I) - \tilde{\theta}_I \\ f(t_I) - \tilde{f}_I \\ t_I - \tilde{t}_I \\ t_F - \tilde{t}_F \end{pmatrix} = 0$$

Then taking $\Phi = \varphi + \nu_i \psi_i$, the transversality conditions for the final costates become

$$\lambda_i^*(t_F) = \begin{cases} -1 & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i = 2, \ldots, 6 \end{cases} \tag{31}$$

while all the initial costates are free, and the value of the Hamiltonian is also free (equal to some component of ν). Since the costates $(\lambda_3, \lambda_4, \lambda_5)$ vanish at the final times, they vanish at all time. The Hamiltonian along the optimal trajectory is $H^* = 1 - \|p\| + h/r^2\lambda_6$.

The costate equations reduce to

$$\dot{\lambda}_i = -\frac{dc}{d\alpha_i}\lambda_6 - \lambda_1 \frac{dF_{11}}{d\alpha_i}u_1 - \lambda_2\left(\frac{dF_{21}}{d\alpha_i}u_1 + \frac{dF_{22}}{d\alpha_i}u_2\right) - \lambda_6\left(\frac{dF_{61}}{d\alpha_i}u_1 + \frac{dF_{62}}{d\alpha_i}u_2\right), \qquad i = 1,2,6$$

with

$$u = -\frac{p}{||p||}, \quad p = \begin{cases} p_1 &= F_{11}\lambda_1 + F_{21}\lambda_2 + F_{61}\lambda_6 \\ p_2 &= F_{22}\lambda_2 + F_{62}\lambda_6 \end{cases}$$

At the final time, the transversality conditions yield to

$$p_2(\alpha(t_F), \lambda(t_F)) = F_{22}(\alpha(t_F))\lambda_2(t_F) + F_{62}(\alpha(t_F))\lambda_6(t_F) = 0$$

hence $u = (1, 0, 0)$ and

$$\begin{aligned} \dot{\lambda}_1(t_F) &= \tfrac{dF_{11}}{da} \\ \dot{\lambda}_2(t_F) &= \tfrac{dF_{11}}{de} \\ \dot{\lambda}_6(t_F) &= \tfrac{dF_{11}}{df} \end{aligned} \tag{32}$$

Therefore the optimal control ends with a tangential thrust. Is the constant tangential control an actual solution of the optimal control problem? For the control to be always tangential, $p_2(\alpha(t), \lambda(t))$ and its higher directional derivatives should vanish along the trajectory, and in particular, at the final time. We start by computing

$$\frac{dp_2}{dt} = \dot{F}_{22}\lambda_2 + \dot{F}_{62}\lambda_6 + F_{22}\dot{\lambda}_2 + F_{62}\dot{\lambda}_6$$

At the final time, the first two terms vanish because of Eq. 31; the second two terms also vanish

$$F_{22}\frac{dF_{11}}{de} + F_{62}\frac{dF_{11}}{da} = \cdots = 0 \tag{33}$$

where we used Eq. 32 and the formulas in Table 3. After some algebra, the third derivative becomes:

$$\frac{d^2 p_2}{dt^2}(t_F) = \cdots = -\frac{h}{r^2}\frac{2a^2\,(e\cos f + 1)}{\mu(e^2 + 2e\cos f + 1)} \leq 0 \tag{34}$$

Therefore the non-tangent component of the control does not vanish along the trajectory, but it approaches 0 with zero derivative and negative curvature.

COMPUTING THE OPTIMAL CONTROL : LINEARIZED SYSTEM

The optimal controls in Table 2 and Eq. 48 are provided open-loop, since the primer vector is only known once the state and costate equations are integrated to solve the boundary constraints. The necessary conditions transform an infinite-dimension problem (where the unknown is the optimal control function) to a finite-dimension two-point boundary value problem:

$$f(y) = 0 \qquad y, f(y) \in \mathbb{R}^q \tag{35}$$

In general, $y = (\alpha_I, \lambda_I, t_F)$ and $f(y)$ is the set of boundary and transversality conditions (Eq. 16, Eq.18, Eq.19)*, and the final states and costates are considered functions of y, i.e. $\alpha_F = \alpha_F(y), \lambda_F = \lambda_F(y)$. The solution to the system can be found numerically using Newton's method, which requires the Jacobian matrix df/dy and hence the derivatives

*But number of equations and unknowns can be reduced using some of the boundary conditions. Also, since we consider autonomous systems, we can assume $t_I = 0$.

$$\frac{d\alpha_F}{d\alpha_I}, \frac{d\alpha_F}{d\lambda_I}, \frac{d\lambda_F}{d\alpha_I}, \frac{d\lambda_F}{d\lambda_I} \tag{36}$$

$$\frac{d\alpha_F}{dt_F}, \frac{d\lambda_F}{dt_F} \tag{37}$$

The last two derivatives (Eq. 37) are found evaluating the r.h.s. of Eq. 2 and 25. The other derivatives (Eq. 36) are computed integrating the state transition matrix:

$$\frac{d}{dt}\left(\begin{bmatrix} \frac{d\alpha(t)}{d\alpha_I} & \frac{d\alpha(t)}{d\lambda_I} \\ \frac{d\lambda(t)}{d\alpha_I} & \frac{d\lambda(t)}{d\lambda_I} \end{bmatrix}\right) = \begin{bmatrix} \frac{d\dot{\alpha}(t)}{d\alpha} & \frac{d\dot{\alpha}(t)}{d\lambda} \\ \frac{d\dot{\lambda}(t)}{d\alpha} & \frac{d\dot{\lambda}(t)}{d\lambda} \end{bmatrix} \begin{bmatrix} \frac{d\alpha(t)}{d\alpha_I} & \frac{d\alpha(t)}{d\lambda_I} \\ \frac{d\lambda(t)}{d\alpha_I} & \frac{d\lambda(t)}{d\lambda_I} \end{bmatrix}, \qquad \begin{bmatrix} \frac{d\alpha(t_I)}{d\alpha_I} & \frac{d\alpha(t_I)}{d\lambda_I} \\ \frac{d\lambda(t_I)}{d\alpha_I} & \frac{d\lambda(t_I)}{d\lambda_I} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{38}$$

where the elements of the Jacobian in eq. 38 are:

$$\begin{aligned}
\frac{d\dot{\alpha}_i}{d\alpha_j} &= \frac{dc_i}{d\alpha_j} + \left(\frac{dF_{ik}}{d\alpha_j}u_k + F_{ik}\frac{du_k}{d\alpha_j}\right) \\
\frac{d\dot{\alpha}_i}{d\lambda_j} &= F_{ik}\frac{du_k}{d\lambda_j} \\
\frac{d\dot{\lambda}_i}{d\alpha_j} &= -\lambda_k\left(\frac{d^2F_{kl}}{d\alpha_i d\alpha_j}u_l + \frac{dF_{kl}}{d\alpha_i}\frac{du_l}{d\alpha_j} + \frac{dc_k}{d\alpha_i d\alpha_j}\right) \\
\frac{d\dot{\lambda}_i}{d\lambda_j} &= -\left(\frac{dF_{jl}}{d\alpha_i}u_l + \frac{dc_j}{d\alpha_i}\right) - \lambda_k\frac{dF_{kl}}{d\alpha_i}\frac{du_l}{d\lambda_j}
\end{aligned}$$

The second derivatives of c_i and F_{ij} are provided in the following section for Keplerian and Equinoctial elements; while the derivatives of the optimal controls in Table 2 are, for case (I),(III), and (IV):

$$\begin{aligned}
\frac{du_m}{d\alpha_j} &= \frac{du_m}{dp_n}\lambda_s\frac{dF_{sn}}{d\alpha_j} \\
\frac{du_m}{d\lambda_j} &= \frac{du_m}{dp_n}F_{jn}
\end{aligned} \tag{39}$$

with

$$\frac{du_m}{dp_n} = \begin{cases} \delta_{mn} & \text{if} \quad u_m = -p_m \\ \left(\frac{p_m p_n}{||p||^3} - \frac{\delta_{mn}}{||p||}\right) & \text{if} \quad u_m = -\frac{p_m}{||p||} \end{cases} \tag{40}$$

For case II in Table 2, the derivatives cannot be computed at switch times. In this case, smoothing techniques are sometimes implemented[39] to make the control differentiable: a family of continuos controls $\tilde{u}$ (parametrized by $\epsilon \in \mathbb{R}$) is introduced with

$$\lim_{\epsilon \to 0} \tilde{u}(p, \epsilon) = u(p)$$

for every $p, ||p|| \neq 1$. Then a family of control problems are solved using continuation methods, starting from a value of ϵ for which the solution is easy to compute. The derivatives of the $\tilde{u}$ with respect to p can then be used in Eq.39 .

Depending on the continuation technique, the additional derivatives might also be needed:

$$\frac{d\alpha_F}{d\epsilon}, \frac{d\lambda_F}{d\epsilon} \tag{41}$$

Eq. 41 are computed integrating

$$\frac{d}{dt}\left(\begin{array}{c}\frac{d\alpha(t)}{d\epsilon}\\ \frac{d\lambda(t)}{d\epsilon_I}\end{array}\right)=\left[\begin{array}{cc}\frac{d\dot{\alpha}(t)}{d\alpha} & \frac{d\dot{\alpha}(t)}{d\lambda}\\ \frac{d\dot{\lambda}(t)}{d\alpha} & \frac{d\dot{\lambda}(t)}{d\lambda}\end{array}\right]\left(\begin{array}{c}\frac{d\alpha(t)}{d\epsilon}\\ \frac{d\lambda(t)}{d\epsilon}\end{array}\right)+\left(\begin{array}{c}\frac{d\dot{\alpha}(t)}{d\epsilon}\\ \frac{d\dot{\alpha}(t)}{d\epsilon_I}\end{array}\right),\qquad\left(\begin{array}{c}\frac{d\alpha(t)}{d\epsilon}\\ \frac{d\lambda(t)}{d\epsilon_I}\end{array}\right)=\left(\begin{array}{c}0\\ 0\end{array}\right) \tag{42}$$

with

$$\begin{aligned}\frac{d\dot{\alpha}_i}{d\epsilon} &= \frac{dF_{ik}}{d\alpha_j}\frac{d\tilde{u}_k}{d\epsilon}\\ \frac{d\dot{\lambda}_i}{d\epsilon} &= -\lambda_k\frac{dF_{kl}}{d\alpha_i}\frac{d\tilde{u}_l}{d\epsilon}\end{aligned}$$

Computing the derivatives

The formulas in this paper require the first and second derivatives of F and c with respect to α. In particular, the costate equations Eq. 25 include first derivatives, while the linearized system Eq. 38 includes first and second derivatives. In this section, Cartesian tensor notation and logarithmic derivatives are used to derive all the derivatives.

In this paper, logarithmic differentiation is represented with the operator Δ , and is defined for a scalar function $f(\alpha) \neq 0$ as the vector of components

$$(\Delta f)_i = \frac{1}{f}\frac{df}{d\alpha_i} \tag{43}$$

Logarithmic differentiation is useful when deriving the derivative of complicated functions, thanks to the properties:

$$\Delta(Kf) = \Delta f, \quad \Delta\left(f^K\right) = K\Delta y \quad \Delta(f^{(1)}f^{(2)}\dots f^{(K)}) = \Delta f^{(1)} + \Delta f^{(2)} + \dots \Delta f^{(K)} \tag{44}$$

where $K \in \mathbb{R}$. Using eq. 43, the first and second derivatives of f can be written as

$$\frac{df}{d\alpha_i} = f\Delta f_i \qquad \frac{d^2 f}{d\alpha_{ij}} = f\left(\frac{d\Delta f_i}{d\alpha_j} + \Delta f_i \Delta f_j\right) \tag{45}$$

For a more generic generic function

$$y(\alpha) = f(\alpha)g(\alpha), \qquad f(\alpha) \neq 0$$

where $g(\alpha) = 0$ for some α, Eq 45 is modified to

$$\frac{dy}{d\alpha_i} = y\Delta f_i + f\frac{dg}{d\alpha\,i} \tag{46}$$

$$\frac{d^2 y}{d\alpha_i d\alpha_j} = y\left(\frac{d\Delta y_i}{d\alpha_j} + \Delta y_i \Delta y_j\right) + f\left(\frac{d^2 g}{d\alpha_j d\alpha_i} + \Delta y_i\frac{dg}{d\alpha_j} + \Delta y_j\frac{dg}{d\alpha_i}\right) \tag{47}$$

Eq. 46-47 are used compute the first and second derivative of all the components of $F(\alpha)$ or $c(\alpha)$, which are typically a product of multiple functions $f(\alpha)$ (such as radius, velocity, etc) with a simple function $g(\alpha)$. Table 3 and 3 show the functions $f, \Delta f_i, \frac{d\Delta f_i}{d\alpha_j}, \frac{dg_i}{d\alpha_j}, \frac{d^2 g}{d\alpha_i d\alpha_j}$ for the components of F and c for Keplerian elements and TNH frame, and Equinoctial elements in RSH frame, respectively. Their derivation is simple thanks to the properties 44. Using Cartesian tensor notations, the formulas are compact and easy to code through nested loops.

The derivatives of F for other local frames are computed using the chain rule on Eq. 9 :

$$
\begin{aligned}
\frac{dF_{ij}^{loc1}}{d\alpha_m} &= \frac{dF_{ik}^{loc2}}{d\alpha_m} A_{kj}^{loc1\to loc2} + F_{ik}^{loc2} \frac{dA_{kj}^{loc1\to loc2}}{d\alpha_m} \\
\frac{dF_{ij}^{loc1}}{d\alpha_m d\alpha_n} &= \frac{dF_{ik}^{loc2}}{d\alpha_m d\alpha_n} A_{kj}^{loc1\to loc2} + \frac{dF_{ik}^{loc2}}{d\alpha_m} \frac{dA_{kj}^{loc1\to loc2}}{d\alpha_n} + \frac{dF_{ik}^{loc2}}{d\alpha_n} \frac{dA_{kj}^{loc1\to loc2}}{d\alpha_m} + F_{ik}^{loc2} \frac{dA_{kj}^{loc1\to loc2}}{d\alpha_m d\alpha_n}
\end{aligned}
$$

Numerical example

This section presents a numerical example of the semi-major axis maximization problem discussed previously. The initial conditions are those of a GTO transfer, while the final time is 10 days. Figure 2 (A) shows the spacecraft trajectory, while Figure 2 (B) shows the history of the costates $\lambda_1, \lambda_2, \lambda_6$ (the others being zero). The close-up shows that the final costates satisfy the transversality conditions Eq. 31. Figure 2 (C) shows the sami-major axis as function of time. Figure 2 (D) shows the components of thrust. The second component is different from zero, although it approaches zero at the final time with zero first derivatives and positive second derivative, as predicted by Eq. 34 (note that the thrust is opposite to the primer vector). The thrust vector oscillates of about 5 degree from the tangential direction.

Figure 3 shows the trajectory, the costate history, the semi-major axis history, and the thrust components for a second solution. The close-up shows that the transversality conditions are also satisfied. In this case, however, the optimal control direction diverges significantly from the tangential direction.

Table 5 shows the final semi-major axis for the first and second case, compared to the final semi-major axis obtained when a constant tangential control is applied. As expected, the difference between the three cases is very small (less than 1% in the change of the final semi-major axis); it is interesting to observe that implementing the necessary conditions can actually result in a worse solution than that obtained with the simplified feedback law.

EFFECT OF THE MASS EQUATION AND OF PERTURBING FORCES

This section briefly discusses the general effects of including the mass equations and additional perturbations. For a more detailed treatment on perturbing forces equinoctial elements see for instance.[18] Consider the equations of motion:

$$
\begin{aligned}
\dot{\alpha}_i &= F_{ij} \left(u_j \tfrac{T_{max}(\alpha,t)}{M} + g_j(\alpha,t) \right) + c_i(\alpha) \\
\dot{M} &= -\tfrac{T_{max}(\alpha,t)}{v_e} \|u\|
\end{aligned}
$$

where $\|u\| \leq 1$, $T_{max}(\alpha,t) \in \mathbb{R}^+$ is the maximum available thrust (it may depends on the distance form the Sun or on the presence of eclipses), and $g_i(\alpha,t)$ is a generic perturbing acceleration. We consider optimization problem in the Meyer form, such as maximizing the final mass $\varphi = -kM(t_F)$ and minimizing the final time $\varphi = t_F$. The Hamiltonian is now time dependent, and becomes:

$$
H(\alpha,u,\lambda,t) = \lambda_i \left(c_i(\alpha) + F_{ij}(\alpha) u_j \frac{T_{max}(\alpha,t)}{M} + F_{ij}(\alpha) g_j(\alpha,t) \right) - \lambda_M \frac{T_{max}(\alpha,t)}{v_e}
$$

where the multiplier λ_M is associated to the mass equation. Minimizing H by using Lemma 1 and 2 yields to

$y(\alpha)$	$\frac{dy}{d\alpha_i} = y\Delta f_i + f\frac{dg}{d\alpha\, i}$			$\frac{d^2y}{d\alpha_i d\alpha_j} = y\left(\frac{d\Delta f_i}{d\alpha_j} + \Delta f_i \Delta f_j\right) + f\left(\frac{d^2 g}{d\alpha_j d\alpha_i} + \Delta f_i \frac{dg}{d\alpha_j} + \Delta f_j \frac{dg}{d\alpha_i}\right)$	
	f	Δf_i	$\frac{dg}{d\alpha\, i}$	$\frac{d\Delta f_i}{d\alpha_j}$	$\frac{d^2 g}{d\alpha_j d\alpha_i}$
h	h	$\eta_i := \frac{\delta_{1i}}{2a} - \frac{e\delta_{2i}}{(1-e^2)}$	$\emptyset$	$H_{ij} := -\frac{\delta_{1i}\delta_{1j}}{2a^2} + -\frac{(1+e^2)}{(1-e^2)^2}\delta_{2i}\delta_{2j}$	$\emptyset$
r	r	$\rho_i := 2\eta_i + +\frac{e\sin f\delta_{6i} - \cos f\delta_{2i}}{1+e\cos f}$	$\emptyset$	$R_{ij} := \left(\frac{\cos f}{1+e\cos f}\right)^2 \delta_{2i}\delta_{2j} + +\frac{\sin f(\delta_{2i}\delta_{6j}+\delta_{6i}\delta_{2j})}{(1+e\cos f)^2} + \frac{e(e+\cos f)\delta_{6i}\delta_{6j}}{(1+e\cos f)^2} + 2H_{ij}$	$\emptyset$
v	v	$\nu_i := \frac{\mu}{v^2}\left(\frac{\delta_{1i}}{2a^2} - \frac{\rho_i}{r}\right)$	$\emptyset$	$V_{ij} := -2\nu_i\nu_j + -\frac{\mu}{v^2}\left(\frac{\delta_{1i}\delta_{1j}}{a^3} + \frac{R_{ij} - \rho_i\rho_j}{r}\right)$	$\emptyset$
F_{11}	$\frac{2a^2 v}{\mu}$	$\frac{2\delta_{1i}}{a} + \nu_i$	$\emptyset$	$V_{ij} - \frac{2\delta_{1i}\delta_{1j}}{a^2}$	$\emptyset$
F_{21}	$\frac{2}{v}$	$-\nu_i$	$\delta_{2i} - \delta_{6i}\sin f$	$-V_{ij}$	$-\delta_{6i}\delta_{6j}\cos f$
F_{22}	$-\frac{r}{av}$	$\rho_i - \nu_i - \frac{\delta_{1i}}{a}$	$\delta_{i6}\cos f$	$R_{ij} - V_{ij} + \frac{\delta_{1i}\delta_{1j}}{a^2}$	$-\delta_{i6}\delta_{j6}\sin f$
F_{33}	$\frac{r}{h}$	$\rho_i - \eta_i$	$-\delta_{i5}\sin\theta$	$R_{ij} - H_{ij}$	$-\delta_{i5}\delta_{j5}\cos\theta$
F_{43}	$\frac{r}{h\sin i}$	$\rho_i - \eta_i - \delta_{i3}\cot i$	$\delta_{i5}\cos\theta$	$\delta_{i3}\delta_{j3}\csc^2 i + R_{ij} - H_{ij}$	$-\delta_{i5}\delta_{j5}\sin\theta$
F_{53}	$-\frac{r}{h\sin i}$	*same as* F_{43}	$-\delta_{i3}\sin\theta\sin i + +\delta_{i5}\cos\theta\cos i$	*same as* F_{43}	$-(\delta_{i3}\delta_{j3} + \delta_{i5}\delta_{j5})\sin\theta\cos i + -\cos\theta\sin i(\delta_{i3}\delta_{j5} + \delta_{i5}\delta_{j3})$
F_{61}	$-\frac{2}{ev}$	$-\nu_i - \frac{\delta_{i2}}{e}$	$\delta_{i6}\cos f$	$-V_{ij} + \frac{\delta_{2i}\delta_{2j}}{e^2}$	$-\delta_{i6}\delta_{j6}\sin f$
[a] F_{62} $\left\{ -\frac{2}{v} \right.$	$-\frac{2}{v}$	$-\nu_i$	$\emptyset$	$-V_{ij}$	
$\left. -\frac{r\cos f}{vea} \right.$	$-\frac{r\cos f}{vea}$	$\rho_i - \nu_i - \frac{\delta_{1i}}{a} - \frac{\delta_{2i}}{e}$	$-\delta_{i6}\sin f$	$R_{ij} - V_{ij} + \frac{\delta_{1i}\delta_{1j}}{a^2} + \frac{\delta_{2i}\delta_{2j}}{e^2}$	$-\delta_{i6}\delta_{j6}\cos f$
$c_5 = c_6$	h/r^2	$\eta_i - 2\rho_i$	$\emptyset$	$H_{ij} - 2R_{ij}$	$\emptyset$

Table 3. Keplerian elements,

[a] To get the derivative, sum the derivatives of the two terms

$y(\alpha)$	$\frac{dy}{d\alpha_i} = y\Delta f_i + f\frac{dg}{d\alpha\, i}$			$\frac{d^2y}{d\alpha_i d\alpha_j} = y\left(\frac{d\Delta f_i}{d\alpha_j} + \Delta f_i \Delta f_j\right) + f\left(\frac{d^2 g}{d\alpha_j d\alpha_i} + \Delta f_i \frac{dg}{d\alpha_j} + \Delta f_j \frac{dg}{d\alpha_i}\right)$	
	f	Δf_i	$\frac{dg}{d\alpha\, i}$	$\frac{d\Delta f_i}{d\alpha_j}$	$\frac{d^2 g}{d\alpha_j d\alpha_i}$
h	h	$\eta_i := \frac{\delta_{1i}}{2a} +$ $-\delta_{2i}\frac{P_1}{1-P_1^2-P_2^2} +$ $-\delta_{3i}\frac{P_2}{1-P_1^2-P_2^2}$	$\emptyset$	$H_{ij} := -\frac{\delta_{2i}\delta_{2j}\left(1+P_1^2-P_2^2\right)}{(1-P_1^2-P_2^2)^2} +$ $-\frac{\delta_{3i}\delta_{3j}\left(1-P_1^2+P_2^2\right)}{(1-P_1^2-P_2^2)^2} - \frac{\delta_{1i}\delta_{1j}}{2a^2} +$ $+\frac{(\delta_{2i}\delta_{3j}+\delta_{2j}\delta_{3i})2P_1P_2}{(1-P_1^2-P_2^2)^2}$	$\emptyset$
A	A	$\rho_i := \frac{\delta_{2i}\sin L}{A} +$ $+\frac{\delta_{3i}\cos L}{A} +$ $+\left(\frac{P_1\cos L}{A} +\right.$ $\left.-\frac{P_2\sin L}{A}\right)\delta_{6i}$	$\emptyset$	$R_{ij} := \frac{\cos L\left(\delta_{6i}\delta_{2j}+\delta_{2i}\delta_{6j}\right)}{A} +$ $-\frac{\sin L\left(\delta_{6i}\delta_{3j}+\delta_{6j}\delta_{3i}\right)}{A} +$ $-\frac{\delta_{6i}\delta_{6j}(P_1\sin L+P_2\cos L)}{A} - \rho_i\rho_j$	$\emptyset$
			$\delta_{2i}\sin L - \delta_{3i}\cos L+$ $+\delta_{6i}\,(P_1\cos L+$ $+P_2\sin L)$		
B	B	$\nu_i := \eta_i - \rho_i$	$\emptyset$	$H_{ij} - R_{ij}$	$\emptyset$
F_{11}	$\frac{2a^2}{h}$	$\frac{2\delta_{1i}}{a} - \eta_i$	$\delta_{3i}\sin L - \delta_{2i}\cos L+$ $+\delta_{6i}\,(P_2\cos L+$ $+P_1\sin L)$	$-\frac{2\delta_{1i}\delta_{1j}}{a^2} - H_{ij}$	$\delta_{6i}\delta_{6j}\,(P_1\cos L - P_2\sin L)$ $+\,(\delta_{3i}\delta_{6j} + \delta_{6i}\delta_{3j})\cos L$ $+\,(\delta_{2i}\delta_{6j} + \delta_{6i}\delta_{2j})\sin L$
F_{12}	$\frac{2a^2}{h}A$	$\frac{2\delta_{1i}}{a} - \eta_i + \rho_i$	$\emptyset$	$-\frac{2\delta_{1i}\delta_{1j}}{a^2} - H_{ij} + R_{ij}$	$\emptyset$
F_{21}	$-AB$	$\rho_i + \nu_i$	$-\delta_{6i}\sin L$	$R_{ij} + V_{ij}$	$-\delta_{6i}\delta_{6j}\cos L$
F_{22}	B	ν_i	$\delta_{6i}\cos L(1+A)+$ $+\delta_{2i} + A\rho_i\sin L$	V_{ij}	$-\delta_{6i}\delta_{6j}(1+A)\sin L$ $+(\rho_i\delta_{6j} + \rho_j\delta_{6i})A\cos L$ $+(\rho_i\rho_j + R_{ij})A\sin L$
F_{23}	BP_2	$\nu_i + \frac{\delta_{3i}}{P_2}$	$\delta_{5i}\sin L - \delta_{4i}\cos L+$ $+\delta_{6i}\,(Q_2\cos L+$ $+Q_1\sin L)$	$V_{ij} - \frac{\delta_{3i}\delta_{3j}}{P_2^2}$	$(\delta_{5i}\delta_{6j} + \delta_{6i}\delta_{5j})\cos L+$ $+(\delta_{4i}\delta_{6j} + \delta_{4i}\delta_{6j})\sin L+$ $+\delta_{6i}\delta_{6j}(Q_1\cos L - Q_2\sin L)$
F_{31}	AB	$\rho_i + \nu_i$	$\delta_{6i}\cos L$	$R_{ij} + V_{ij}$	$-\delta_{6i}\delta_{6j}\sin L$
F_{32}	B	ν_i	$-\delta_{6i}\sin L(1+A)+$ $+\delta_{3i} + \rho_i A\cos L$	V_{ij}	$-\delta_{6i}\delta_{6j}(1+A)\cos L$ $-(\rho_i\delta_{6j} + \rho_j\delta_{6i})A\sin L$ $+(\rho_i\rho_j + R_{ij})A\cos L$
F_{33}	BP_1	$\nu_i + \frac{\delta_{2i}}{P_1}$	*see* F_{23}	$V_{ij} - \frac{\delta_{2i}\delta_{2j}}{P_1^2}$	*see* F_{23}
F_{43}	$\frac{B\left(1+Q_1^2+Q_2^2\right)}{2}$	ν_i+ $+\frac{2Q_1\delta_{4i}}{1+Q_1^2+Q_2^2}+$ $+\frac{2Q_2\delta_{5i}}{1+Q_1^2+Q_2^2}$	$\delta_{6i}\cos L$	$V_{ij} - \frac{2Q_1Q_2\left(\delta_{4i}\delta_{5j}+\delta_{5i}\delta_{4j}\right)}{\left(1+Q_1^2+Q_2^2\right)^2}+$ $+2\delta_{5i}\delta_{5j}\frac{1+Q_1^2-Q_2^2}{\left(1+Q_1^2+Q_2^2\right)^2}+$ $+2\delta_{4i}\delta_{4j}\frac{1-Q_1^2+Q_2^2}{\left(1+Q_1^2+Q_2^2\right)^2}$	$-\delta_{6i}\delta_{6j}\sin L$
F_{53}	*see* F_{43}	*see* F_{43}	$-\delta_{6i}\sin L$	*see for* F_{43}	$-\delta_{6i}\delta_{6j}\cos L$
F_{63}	B	ν_i	$\delta_{2i}\sin L - \delta_{3i}\cos L+$ $+\delta_{6i}\,(P_1\cos L+$ $+P_2\sin L)$	V_{ij}	$\delta_{6i}\delta_{6j}\,(P_1\cos L - P_2\sin L)$ $+\,(\delta_{2i}\delta_{6j} + \delta_{6i}\delta_{2j})\cos L$ $+\,(\delta_{3i}\delta_{6j} + \delta_{6i}\delta_{3j})\sin L$
c_6	$(Bh)^{-1}$	$-\nu_i - \eta_i$	$\emptyset$	$-V_{ij} - H_{ij}$	$\emptyset$

Table 4. Equinoctial elements, polar frame

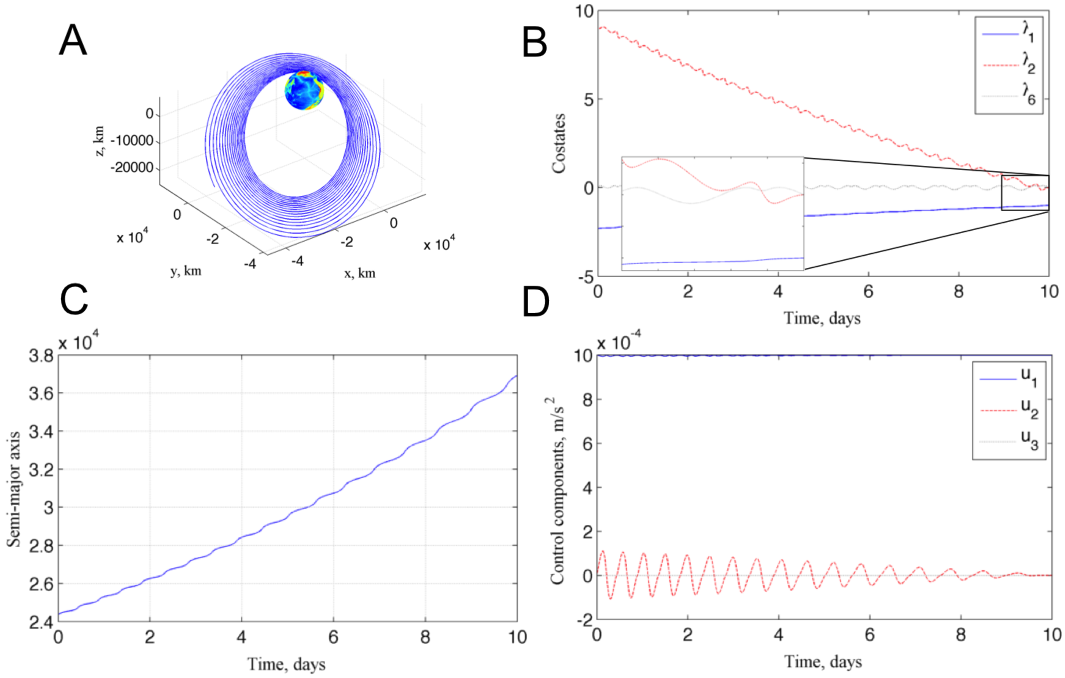

Figure 2. Trajectory, costates, semi-major axis, and thrust components for a solution to the maximum final semi-major axis problem

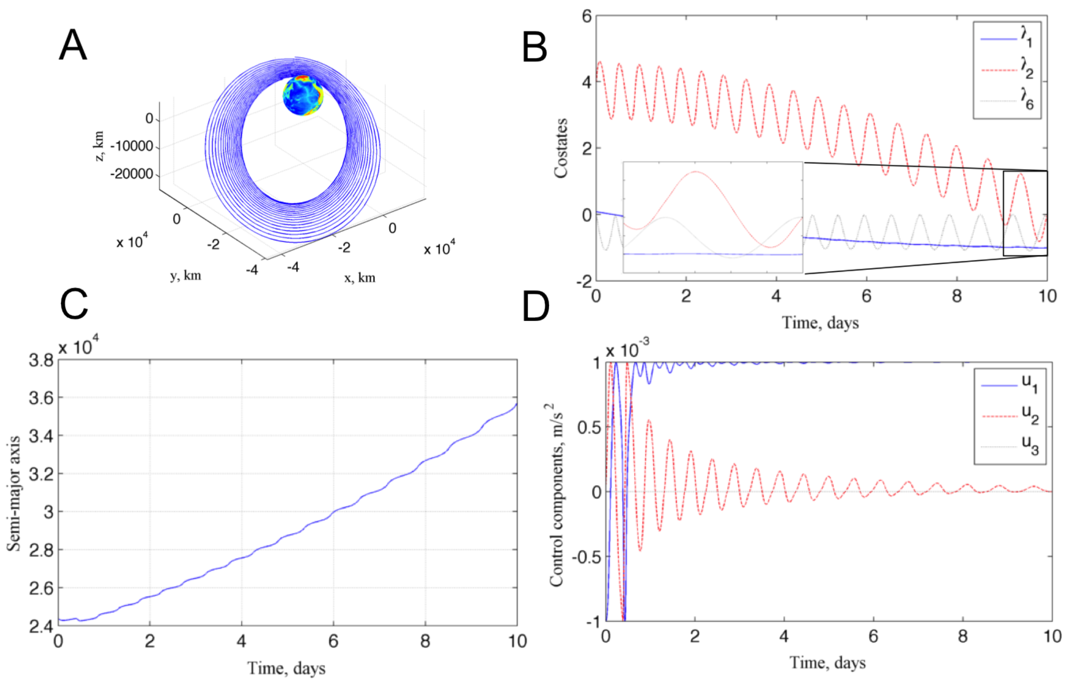

Figure 3. Trajectory, costates, semi-major axis, and thrust components for a second solution to the maximum final semi-major axis problem

	Open-loop (I)	Open-loop (II)	Feedback (tangential)
Final Semi-major axis	$36,919$ km	$35,962$ km	$36,903$ km

Table 5. Comparison between tangent feedback and optimal solutions A,B, starting from a GTO (a0=24371 km)

$$u^*(t) = \begin{cases} \frac{p}{||p||} & \text{if} \quad ||p(\alpha,\lambda)|| > -\lambda_M \frac{M}{v_e} \\ 0 & \text{if} \quad ||p(\alpha,\lambda)|| < -\lambda_M \frac{M}{v_e} \end{cases} \tag{48}$$

where the same definition of the primer vector Eq. 24 holds:

$$p_i(\alpha,\lambda) = \lambda_i F_{ij}(\alpha)$$

The equations above shows that even when including perturbations and the mass equation, the optimal control direction is again opposite to the primer vector. Of course, since the equations of λ are different, the time-dependent weights will be different.The costate equations become

$$\begin{array}{rcl} \dot{\lambda}_i & = & -\lambda_k \left(\frac{dc_k}{d\alpha_i} + \frac{\partial F_{kj}}{\partial \alpha_i} \left(u_j \frac{T_{max}}{M} + g_j \right) + F_{kj} \left(\frac{u_j}{M} \frac{\partial T_{max}}{\partial \alpha_i} + \frac{\partial g_j}{\partial \alpha_i} \right) \right) - \lambda_M \frac{T_{max}(\alpha,t)}{v_e} \\ \dot{\lambda}_M & = & -\frac{T_{max}}{M^2} \lambda_k F_{kj} u_j \end{array}$$

Applying the optimal control to the last equation yields to

$$\dot{\lambda}_M = \begin{cases} -\frac{T_{max}}{M^2} ||p|| & \text{if} \quad ||p(\alpha,\lambda)|| > -\lambda_M \frac{M}{v_e} \\ 0 & \text{if} \quad ||p(\alpha,\lambda)|| < -\lambda_M \frac{M}{v_e} \end{cases}$$

For minimum propellant mass problem, the transversality conditions related to the final mass yields to

$$\lambda_M(t_f) = \frac{d\varphi}{dM(t_f)} = -k$$

It is well known that , since $\dot{\lambda}_M \leq 0$, the transverality condition is always satisfied for some k by picking $\lambda_M(t_I) < 0$. For minimum-time problem, the final mass is a free parameter and does not appear in the merit function, yielding to $\lambda_M(t_F) = 0$ and to $\lambda_M(t) > 0 \geq -\frac{v_e}{M\|p\|}$ for any t; therefore there are no coast arcs.

CONCLUSION

This paper contains theoretical developments on the application of optimal control on Gauss equation. In particular, the optimal control solutions of a large class of problem are shown to follow a primer vector, equivalent to Lawden's primer vector in Cartesian coordinates; and optimal control primitives are defined as special feedback functions, that the optimal control balances with the costates acting as time-dependent weights.

The specific example of Keplerian and equinoctial elements are then considered. In Keplerian element, it is proven that the tangential control law (corresponding to the optimal control primitive associated to the semi-major axis) is not a the optimal control that maximizes the final semi-major axis, although it is confirmed to be a good approximation. Finally, the first and second derivatives of the coefficient of Gauss equations are provided in a compact form, using to the Cartesian tensor notation and logarithmic differentiation.

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NOTATION

a Semi-major axis

e Eccentricity

f True anomaly (when Keplerian element).Otherwise, generic function.

i Inclination (when Keplerian element).

Ω Argument of the ascending node

ω Argument of the pericenter

θ Argument of latitude

L True longitude

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