4th IAA Planetary Defense Conference – PDC 2015
13-17 April 2015, Frascati, Roma, Italy

IAA-PDC-15-02-11

The linear method for impact probability estimation using a curvilinear coordinate system

D. E. Vavilov[1], Yu. D. Medvedev[2]
[1][2]*Institute of Applied Astronomy of Russian Academy of Science, Russia, St. Petersburg, Kutuzova emb.10, +7(812)275-10-90,*

Keywords: *impact probability, fast numerical method, orbital coordinate system.*

Abstract

In this paper the linear method of impact probability calculation using a special curvilinear coordinate system associated with the nominal orbit of an asteroid is described. Comparison with a linear method, which uses a Cartesian coordinate system, shows that the described method is more reliable. As etalon values of the impact probabilities we take ones obtained by the Monte Carlo method. Comparison with the LOV method shows that the LOV method gets more accurate values of impact probability in cases where there are close approaches to major planets before the potential collision. However the described method can be considered as a first step in impact probability calculation problem.

1. Introduction

Different methods of impact probability estimation have different limitations and therefore different fields of usage. The Monte Carlo method is the easiest and the most theoretically based method. However the amount of Virtual Asteroids (VA), the orbits of which we have to propagate, directly depends on the impact probability value and can be written as:

$$m = \frac{P_{MC}(1 - P_{MC})}{\sigma_{MC}^2},$$

where σ_{MC} is the error in probability P_{MC}. Thus, in order to calculate an impact probability 10^{-6} with 30% accuracy, it is necessary to propagate approximately 10^7 VA orbits. Thus, the Monte Carlo method can be impractical for low impact probabilities.

A more efficient method is the method called line-of-variation sampling (Milani et al. 2002). In this method, instead of a six-dimensional cloud of VAs, we consider a one-dimensional region – Line of Variations that is hopefully representative of the entire six-dimensional one. Even though interpolation along the LOV is generally more efficient than the Monte Carlo method, we cannot be confident that the LOV approach will detect all potential collisions. Also, it requires one to perform simulations too, but not as many as in the Monte Carlo method.

There are also linear methods (ex. the target plane method) which assume a linear relation between orbital parameters errors at epoch of observations and time of possible collision, therefore the distribution of the orbital parameters errors remains normal. Generally as orbital parameters one use Cartesian coordinates and velocities, however even in 2-body problem the distribution of VAs will not be normal in Cartesian coordinates and velocities if the errors are not small. The distribution of

VAs is mainly along the nominal orbit, and we can't take it into account using a Cartesian coordinate system.
In this work we describe a linear method, which uses a curvilinear coordinate system associated with the nominal orbit of an asteroid (Vavilov, Medvedev, 2015). Also we provide a comparison of the impact probabilities for 14 asteroids with a linear method, which uses a Cartesian coordinate system, and nonlinear methods: sampling of mean motion, LOV sampling and the Monte Carlo one.

2. Curvilinear coordinate system

We've constructed a special curvilinear coordinate system which allows us to take into account the fact that the distribution of VAs is mainly along the nominal orbit. This system is associated with the osculating orbit of an asteroid. First of all, we fix the osculating ellipse of a small body at time t (i.e. the five parameters of the osculating ellipse). The mean anomaly M in the osculating orbit is one of the coordinates of this system. The origin of the spatial coordinates ξ, η is the point on the ellipse corresponding to M. This system is required to be an orthogonal one. The ξ-axis is perpendicular to the plane of the fixed ellipse:

$$\boldsymbol{e_\xi} = (\sin i \sin \Omega, -\sin i \cos \Omega, \cos i),$$

where i is the orbit inclination and Ω the longitude of the ascending node.
This system is schematically illustrated in Fig. 1.

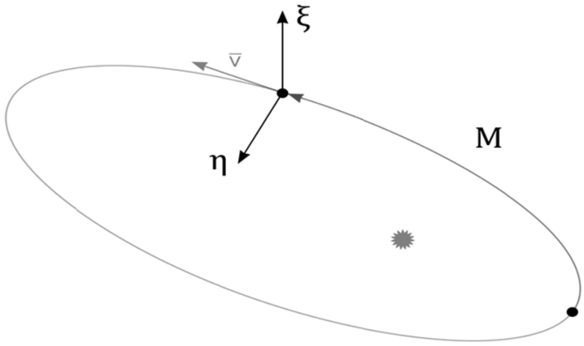

Fig. 1. The (ξ, η, M) coordinate system.

Let us consider in detail how to find the coordinates ξ, η, M of any point in space if its Cartesian coordinates are x, y, z. Let x', y', z' be coordinates of a projection of point x, y, z to the plane of the fixed ellipse. The coordinate ξ is obtained from:

$\xi \boldsymbol{e_\xi} = (x - x', y - y', z - z')$.

Let x_0, y_0 be the coordinates of x', y', z' in the plane of the fixed ellipse, with the origin in the center of the ellipse, and x_1, y_1 be the coordinates of the point on the ellipse corresponding to M, E – eccentric anomaly and a be a semi-major axis of the ellipse. Then

$$\begin{cases} x_1 = a \cos E \\ y_1 = a\sqrt{1 - e^2} \sin E \end{cases}$$

$$\boldsymbol{e_M} = (-\sin M, \cos M).$$

Consequently, the condition $\boldsymbol{e_\eta} \perp \boldsymbol{e_M}$ is:

$$-(a \cos E - x_0) \sin M + (a\sqrt{1 - e^2} \sin E - y_0) \cos M = 0.$$

Since at $M = 0$ and $M = \pi$ the left-hand side of the equation has different signs, hence in both regions $[0, \pi)$ and $[\pi, 2\pi)$ there is at least one root. We choose the root (M) that corresponds to the closest point to (x_0, y_0). Then we calculate x_1, y_1. η we can find from: $\eta \boldsymbol{e_\eta} = (x_1 - x_0, y_1 - y_0)$.

3. Impact probability calculation

In order to calculate the impact probability at time t using the curvilinear coordinate system we need to find the covariance matrix $\mathbf{C}_{\xi\eta M}$ in this system at time t. This matrix is related with the covariance matrix in a Cartesian coordinate system $\mathbf{C}_{xyz}$ by:

$$\mathbf{N}_{\xi\eta M} = \mathbf{C}_{\xi\eta M}^{-1} = \mathbf{Q}^T \mathbf{C}_{xyz}^{-1} \mathbf{Q},$$

where $\mathbf{Q}$ is the transfer matrix:

$$Q = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \cdots & \frac{\partial\xi}{\partial\dot{z}} \\ \vdots & \ddots & \vdots \\ \frac{\partial\dot{M}}{\partial x} & \cdots & \frac{\partial\dot{M}}{\partial\dot{z}} \end{pmatrix}.$$

The impact probability equals:

$$P = \frac{|det\mathbf{N}_{\xi\eta M}|}{(2\pi)^3} \int_{\Theta} e^{-(1/2)\boldsymbol{x}^T \mathbf{N}_{\xi\eta M}\boldsymbol{x}}\, d\boldsymbol{x},$$

where $\boldsymbol{x}$ is a six-dimensional vector of deviations of $(\xi, \eta, \ldots, \dot{M})$ from the nominal values, Θ is a six-dimensional volume of the Earth in $(\xi, \eta, \ldots, \dot{M})$. Note that in $(\dot{\xi}, \dot{\eta}, \dot{M})$ the set Θ is $(-\infty, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty)$ and we can integrate over these components analytically (see Vavilov, Medvedev, 2015). Then we a have 3-dimensional integral, 3-dimensional volume of the Earth Θ' and a 3-dimensional matrix $\mathbf{N}'_{\xi\eta M}$.

In order to define Θ' we modify the introduced system so that Θ' has a sphere-like shape. Let the coordinates of the Earth's center in this system be (ξ_t, η_t, M_t). Then the projection of Θ' on to the coordinate M is the interval $[-\frac{R_\oplus}{V}\dot{M} + M_t, \frac{R_\oplus}{V}\dot{M} + M_t]$, where $R_\oplus$ is the Earth's radius and V is the magnitude of the heliocentric velocity of the asteroid at time t.

Consider the coordinate system $(\xi, \eta, M/d)$, where $d = \frac{\dot{M}}{V}$. In this frame, the projection Θ' on to the (ξ, η) plane is a circle with radius $R_\oplus$ and the projection Θ' on to the coordinate M/d is an interval with semi-length also $R_\oplus$. Since the Earth's radius $R_\oplus \ll 1\, au$, Θ' can be considered as a full sphere in coordinates $(\xi, \eta, M/d)$. In order to find the normal matrix $\mathbf{N}'_{\xi\eta M/d}$ in the coordinates $(\xi, \eta, M/d)$, we need to multiply the elements of the matrix $\mathbf{N}'_{\xi\eta M}$ in the third column and the elements in the third row by d (in this case the element in the third column and third row is multiplied by d^2).

To decrease the time of numerical calculation of the integral, we use a singular decomposition of the matrix $\mathbf{N}'_{\xi\eta M/d} = \mathbf{U} \cdot \mathbf{N}^* \cdot \mathbf{U}^{-1}$, where $\mathbf{N}^*$ is a diagonal 3×3 matrix and $\mathbf{U}$ is an orthogonal matrix. Consider the coordinate system $(\xi^*, \eta^*, M^*/d)$, which is a product of the orthogonal matrix $\mathbf{U}$ and $(\xi, \eta, M/d)$. Since this is an orthogonal transformation, the region Θ' remains a full sphere with the same radius in the frame $(\xi^*, \eta^*, M^*/d)$. This transformation yields the fact that there are no linear correlations between ξ^*, η^* and M^*/d, since $\mathbf{N}^*$ is a diagonal matrix. Then Θ' is replaced with a cube with semi-side $R_\oplus$ (and the same center). Since there are no correlations and the region Θ' is a cube, the three-dimensional integral in coordinates $(\xi^*, \eta^*, M^*/d)$ becomes a multiplication of three one-dimensional Laplace integrals:

$$\int_{-R_\oplus}^{+R_\oplus} e^{-\left(\frac{1}{2}\right)\left(\frac{y^2}{\sigma^2}\right)} dy.$$

We should consider time t close to the time when the distance between the Earth and the nominal asteroid orbit is minimum.

4. Results

To verify this method, we considered impact probabilities for 14 asteroids. We chose these asteroids randomly from the website of the Jet Propulsion Laboratory, NASA, but ensuring impact probabilities more than 10^{-7}. Their orbits were calculated by a method based on an exhaustive search for orbital planes (Bondarenko, Vavilov & Medvedev 2014). The normal matrix at the initial epoch was computed by a differential method. The selected asteroids have different values of orbital inclinations $0.9 < i < 25.1$ and a wide range of eccentricities $0.09 < e < 0.74$. It should be emphasized that for some object available more observations that change the impact probability value, however we can consider these objects as model ones.
The results are represented in Table 1. $P_{\xi\eta M}$ are probabilities calculated by the proposed method in the curvilinear coordinate system. As etalon values of impact probability we consider the values obtained by the Monte Carlo method (P_{MC}). The errors of P_{MC} are given by: $\sigma_{MC} = \sqrt{P_{MC}(1 - P_{MC})}/\sqrt{m}$. To show the advantage of the introduced curvilinear coordinate system we calculated impact probabilities by the linear method using a Cartesian coordinate system (P_{xyz}). The scheme of this method is almost the same with the described above but instead of matrix $\mathbf{N}_{\xi\eta M}$ one should use $\mathbf{C}_{xyz}^{-1}$ and Cartesian coordinates and velocities.
We also calculated the impact probabilities by the method LOV as it described in (Milani et al. 2002). Along Line Of Variations we find the virtual asteroid (VA) which has the deepest close approach to the center of the Earth. Then taking this VA as nominal we compute the impact probability P_1 by the linear method. The one difference with the method, described in (Milani et al. 2002) is that the probability P_1 is calculated by the linear method using a Cartesian coordinate system taking a 6-dimensional integral, instead by the target plane method. But since this VA is not actually the nominal orbit, we correct P_1 for the distance from the nominal to obtain: $P_{LOV} = P_1 e^{-\sigma_\Lambda^2/2}$, where σ_Λ is the σ distance from the nominal asteroid along the LOV.

Table 1. Results.

Object	t	P_{xyz}	$P_{\xi\eta M}$	P_{LOV}	P_{MC}	$3\delta_{MC}$,%
2006 JY26	2073	$1.14 \cdot 10^{-5}$	$1.14 \cdot 10^{-4}$	$6.31 \cdot 10^{-5}$	$5.63 \cdot 10^{-5}$	29.6
2010 UK	2068	$2.67 \cdot 10^{-3}$	$2.64 \cdot 10^{-3}$	$2.79 \cdot 10^{-3}$	$3.07 \cdot 10^{-3}$	23.9
2006 QV89	2019	$2.26 \cdot 10^{-3}$	$2.20 \cdot 10^{-3}$	$2.24 \cdot 10^{-3}$	$1.79 \cdot 10^{-3}$	5.8
2011 AG5	2040	$5.00 \cdot 10^{-3}$	$5.08 \cdot 10^{-4}$	$4.85 \cdot 10^{-4}$	$5.28 \cdot 10^{-4}$	24.3
2007 VK184	2048	$2.91 \cdot 10^{-5}$	$3.01 \cdot 10^{-5}$	$1.11 \cdot 10^{-5}$	$6.18 \cdot 10^{-6}$	32.3
2007 VE191	2015	0	$6.31 \cdot 10^{-4}$	$6.38 \cdot 10^{-4}$	$6.36 \cdot 10^{-4}$	15.8
2008 CK70	2030	$6.42 \cdot 10^{-4}$	$6.41 \cdot 10^{-4}$	$6.14 \cdot 10^{-4}$	$6.43 \cdot 10^{-4}$	15.0
2009 JF1	2022	$6.60 \cdot 10^{-4}$	$6.56 \cdot 10^{-4}$	$6.29 \cdot 10^{-4}$	$7.44 \cdot 10^{-4}$	15.6
2012 MF7	2046	0	$3.96 \cdot 10^{-4}$	$2.94 \cdot 10^{-4}$	$3.11 \cdot 10^{-4}$	25.4
2014 WA	2049	0	$4.52 \cdot 10^{-7}$	$7.37 \cdot 10^{-7}$	$3.17 \cdot 10^{-7}$	75.0
2008 JL3	2027	$4.75 \cdot 10^{-4}$	$4.70 \cdot 10^{-4}$	$4.76 \cdot 10^{-4}$	$2.97 \cdot 10^{-4}$	14.6
2005 BS1	2016	0	$1.48 \cdot 10^{-4}$	$1.50 \cdot 10^{-4}$	$1.45 \cdot 10^{-4}$	16.3
2005 QK76	2030	0	$3.77 \cdot 10^{-5}$	$3.83 \cdot 10^{-5}$	$4.28 \cdot 10^{-5}$	19.8
2007 KO4	2015	0	$3.97 \cdot 10^{-7}$	$6.42 \cdot 10^{-7}$	$7.33 \cdot 10^{-7}$	53.9

'Object' is the asteroid designation, t the year of possible collision, $3\delta_{MC} = 3\sigma_{MC}/P_{MC}$

The table shows that using the curvilinear coordinate system instead of a Cartesian one in a linear method has an advantage. For 6 cases the linear method in Cartesian coordinates didn't find the possible collision while the proposed method obtained the impact probability values close to P_{MC}. However for 2006 JY26 and 2007 VK184 the values $P_{\xi\eta M}$ higher than P_{MC} (2 and 5 times correspondingly). This fact is likely due to the close approaches of the cloud of virtual asteroids with major planets before the time of the potential collision. More importantly, that close approaches has the area of the cloud of VAs, which leads to the collision, while the approaches of the nominal orbit aren't deep.

The impact probabilities obtained by the LOV method are in good agreement with those obtained by the Monte Carlo method. The exception is 2007 VK184. The P_{LOV} value for it is about 2 times higher. This is also due to close approaches, but the effect is less than for the linear methods. This situation is interesting because the VA, which has the deepest close approach along LOV, collides with the Earth ($\sigma_\Lambda = -3.4061, r_{min} = 4529\ km$). Table 1 also shows that for 2014 WA the developed method got a bit closer result to P_{MC}. The P_{LOV} values for these asteroids can probably be corrected by the techniques described in (Milani et al. 2005).

5. Conclusion

To sum up we can say that the developed linear method, using a curvilinear coordinate system, has advantages in comparison with the linear method, using a Cartesian coordinate system. The developed linear method works well enough in cases where there are no deep close approaches of the cloud of VAs to major planets before the potential collision. According to the results the LOV method is more reliable than the developed one, but the developed method requires several orders of magnitude fewer computation time, since we have to propagate the orbit of an asteroid only once. Consequently, this method can be implemented as a first step in impact probability calculation problem.

References

1. Bondarenko Yu. S., Vavilov D. E., Medvedev Yu. D., 2014, Method of Determining the Orbits of the Small Bodies in the Solar System Based on an Exhaustive Search of Orbital Planes // Solar System Research, vol. 48 issue 3, pp. 212-216.
2. Milani A., Chesley S.R., Chodas P.W., Valsecchi G.B., 2002, Asteroid Close Approaches: Analysis and Potential Impact Detection. In Bottke W. F. Jr., Cellino A., Paolicchi P., Binzel R. P., eds, Asteroid III, Space Science Series, Univ. Arizona Press, p. 55-69.
3. Milani A., Chesley S.R., Sansaturio M.E., Tommei G., Valsecch G.B., 2005, Nonlinear impact monitoring: line of variation searches for impactors, Icarus, 173, p. 362-384.
4. Vavilov D. E., Medvedev Yu. D. A fast method for estimation of the impact probability of near-Earth objects // Monthly Notices of the Royal astronomical society, 2015, 446, p. 705-709.