ROBUST RENDEZ-VOUS PLANNING USING THE SCENARIO APPROACH AND DIFFERENTIAL FLATNESS

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There is a growing interest in uncertainty handling in the spacecraft dynamics community. In particular, robust optimization of spacecraft maneuvers is regarded as an important challenge. This paper proposes an optimal control approach for orbital rendez-vous planning under stochastic dynamics and constraints. The method combines differential flatness theory with the scenario approach for optimization under uncertainties. By mapping state and control variables into a set of flat outputs, the enforcement of dynamics equations and boundary conditions is automatically satisfied. The rigorous foundations of the scenario approach lead to a finite-dimensional formulation of the problem which guarantees the feasibility of the solution within an arbitrary portion of the stochastic domain. The methodology is illustrated by means of two case studies involving a rendez-vous in elliptic orbit and a propellantless maneuver using differential drag, respectively.

INTRODUCTION

Uncertainties in the modeling of perturbations like atmospheric drag make orbital dynamics inherently stochastic. Specifically, stochastic space weather proxies like solar and geomagnetic activity are inputs for several atmospheric models, while difficulties in the determination of aerodynamics coefficients are due to complex physical phenomena like the gas-surface interaction, or to uncertainties in the attitude estimation and control.\(^1\)

The robust formulation of orbital maneuvers is gaining increasing interest in the literature. This is specifically the case for rendez-vous maneuvers, which are most often handled by means of over-simplified reference models, e.g., the Hill Clohessy (HC) equations. Luo et al.\(^2\) proposed a hybrid optimal approach for robust rendez-vous, which combines a genetic algorithm and Newton’s method to handle uncertainties in discrete and continuous design variables, respectively. Gao et al.\(^3,4\) investigated the problem of $H_\infty$ control of the perturbed HC equations and, after deriving conditions for admissible controllers, proposed a Lyapunov $H_\infty$ control approach to satisfy multi-objective requirements. Another hybrid optimal control approach based on the stochastic version of the projected gradient was proposed by Gomes et al.\(^5\) This formulation allowed the enforcement of the constraints of the optimization problem to be relaxed in order to accommodate uncertainties. By using genetic algorithm and Monte Carlo simulation in conjunction with stochastic control, Xu et al.\(^6\) developed an approach to generate and track reference maneuvers. A feedback model predictive control strategy for elliptic rendez-vous maneuvers was finally proposed by Deaconu et al.\(^7\)

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This paper focuses on the problem of orbital rendez-vous maneuver planning under uncertainties in the dynamics equations. A novel optimal control approach which combines differential flatness theory with the scenario approach developed by Calafiore and Campi is proposed for this purpose. By mapping state and control variables into a set of flat outputs, the enforcement of dynamics equations and boundary conditions is automatically satisfied. The rigorous foundations of the scenario approach lead to a finite-dimensional formulation of the problem which guarantees the feasibility of the solution within an arbitrary portion of the stochastic domain.

After introducing differential flatness and the scenario approach, the problem of robust maneuver planning is formulated. The proposed methodology is then illustrated by means of two case studies. The first consists of a fully-actuated elliptical rendez-vous maneuver. The second considers a circular orbit and propellantless maneuver control using differential drag. Uncertainties in the aero-dynamic force are considered for both cases. The paper closes with a discussion on the theoretical results against the practical implementation of the planned maneuver.

DIFFERENTIAL FLATNESS

Differential flatness was first introduced by Fliess. The use of differential flatness for the planning of orbital rendez-vous maneuvers was proposed by Louembet. Consider a dynamical system \( \dot{x} = f(x, u) \) with \( n \) states, \( x(t) : \mathbb{R} \to \mathbb{R}^n \), and \( m \leq n \) inputs, \( u(t) : \mathbb{R} \to \mathbb{R}^m \). The system is said flat if a set of \( m \) variables

\[
q = Q(x, u, \dot{u}, \ddot{u}, \ldots),
\]

exists such that

\[
\begin{align*}
x &= X(q, \dot{q}, \ddot{q}, \ldots), \\
u &= U(q, \dot{q}, \ddot{q}, \ldots);
\end{align*}
\]

and \( q \in \mathbb{R}^m \) are referred to as flat outputs.

Flatness is a property of the system. Proving that a system is flat is not necessary an easy task. Linear systems are differentially flat if and only if they are controllable.

The important property of the flatness exploited in this paper is that given a sufficiently smooth trajectory for the flat outputs, \( q(t) \) with \( t \in [0, T] \), the corresponding time evolution of the states and the control necessary to obtain the trajectory are analytically determined by mapping \( q(t) \) through \( X \) and \( U \), respectively.

THE SCENARIO APPROACH

Consider the robust optimization problem

\[
y^* = \arg \left[ \min_y \left( c^T y \right) \right] \quad \text{s.t. :}
\]

\[
g(y, \delta) \leq 0 \quad \forall \delta \in \Delta.
\]

where \( y \in \mathbb{R}^d, \delta \in \Delta, c, \) and \( g \) are the \( d \)-dimensional vector of design variables, a generic random quantity (e.g., a set of random variables and stochastic processes) defined on the sample space \( \Delta \) and provided with probability distribution \( P_\Delta \), a constant vector, and a set of constraints, respectively. The optimal solution of Problem 3 is robust all over the uncertain domain \( \Delta \). In other words, given any sample \( \delta \in \Delta, y^* \) is feasible, i.e., \( g(y^*, \delta) \leq 0 \).
When the event space of $\Delta$ is infinite-dimensional – as it is most often the case – solving Problem 3 can be a real challenge. In this case, because the number of design variables is finite, the problem is called semi-infinite. The scenario approach is a tool that allows to solve a relaxed version of the semi-infinite Problem 3, where the obtained solution is only feasible in a subset $\Delta_\epsilon \in \Delta$ such that $P_{\Delta}(\Delta_\epsilon) \geq 1 - \epsilon$. Here $\epsilon \in (0, 1]$ is referred to as risk parameter.

Consider problem 3 and assume that $g$ is convex with respect to $y$. The scenario approach states that: given a confidence parameter $\beta \in [0, 1)$, a risk $\epsilon$, and $s$ independent instances $(\delta_1, \ldots, \delta_s)$ of $\Delta$ extracted according to $P_{\Delta}$ and such that

$$s \geq \frac{2}{\epsilon} (d - \ln \beta),$$

the solution of the finite-dimensional problem

$$y^* = \arg \left[ \min_y \left( c^T y \right) \right] \quad s.t. :$$

$$g(y, \delta_1) \leq 0$$

$$\vdots$$

$$g(y, \delta_s) \leq 0$$

satisfies all the constraints in $\Delta$ but at most a portion $\epsilon$ with probability $1 - \beta$. Figure 1 provides with a graphical interpretation of this theorem.

One of the most appealing features of the scenario approach, is that the risk $\epsilon$ is selected by the user, so that it can be made as small as desired. Furthermore, since the only requirement for the scenario approach is the convexity with respect to design variables, it has an extremely high level of generality and no requirement exists for the uncertain set.

We note that the confidence parameter, $\beta$, appears as the argument of a logarithm in Equation 4. When $\beta$ approaches 0 its logarithm decreases slowly. For practical purposes, the confidence parameter can be chosen small enough to be neglected, e.g., $\beta = 10^{-7} \Rightarrow -\ln \beta \simeq 16$.

Another very interesting property of the scenario approach is that it does not require a probabilistic characterization of the stochastic sources of the problem. In fact, even though it assumes the existence of a probability distribution $P_{\Delta}$, it does not require its knowledge, but it only requires the realization of a certain number of samples. For this reason, the scenario approach facilitates the inclusion of uncertainty sources of arbitrary nature in the dynamics, e.g., random variables, stochastic processes, and random fields. Assume, for example, that the uncertainty due to the high order harmonics of the gravitational field are included in the rendez-vous problem. Because the variance

![Figure 1. Schematic representation of the scenario approach as illustrated in.](image-url)
of this perturbation decreases with the relative distance and because it also depends on time given a
fixed relative position, this uncertainty should be properly characterized by means of a non-uniform
and non-stationary random field dependent on the relative position and on time. However, with the
scenario approach it is only sufficient to generate a certain number of samples. This can be achieved,
for example, by comparing the specific force of the reference dynamics equations with respect to a
higher precision propagation.

We finally note that the scenario approach does not allow to identify the feasible set \( \Delta_\epsilon \). If needed, a posteriori Monte Carlo evaluation of \( g(y^*, \delta) \) can reveal \( \Delta_\epsilon \).

More advanced and detailed results on the scenario approach are available in references.\(^{10,11,12}\)

**MATHEMATICAL FORMULATION OF ROBUST SPACECRAFT RENDEZ-VOUS**

Consider the robust optimal control formulation of the rendez-vous problem between two space-
craft

\[
x^* = \arg \left[ \min_{\mathbf{x}(t), t \in [0,T]} \left( \max_{\delta \in \Delta} J(x, \delta) \right) \right]
\]

\[
\text{s.t. :}
\]

\[
\dot{x} = f(x, u, \delta) \quad \forall \ t \in [0, T] \wedge \delta \in \Delta
\]

\[
g(x, u, \delta) \leq 0 \quad \forall \ t \in [0, T] \wedge \delta \in \Delta
\]

\[
x(0) = x_0
\]

\[
x(T) = 0,
\]

where \( J : \mathbb{R}^n \times \Delta \rightarrow \mathbb{R} \), \( f \in \mathbb{R}^n \), \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( \delta \in \Delta \), \( T \), and \( x_0 \) are the objective function (convex with respect to \( x \)), the right hand terms of the stochastic dynamics equations, the relative position and velocity of the chaser spacecraft with respect to the local vertical local horizontal (LVLH) frame centered in the target, the control forces per unit mass, a generic uncertain quantity defined on \( \Delta \), the maneuvering time, i.e., the time when rendez-vous conditions are met, and the initial state vector, respectively. \( g \in \mathbb{R}^c \) expresses constraints like the saturation of the control variables.

The objective of Problem 6 is to find a *deterministic trajectory* of the states which is robust against uncertainties in the dynamics, i.e., which could be ideally followed for any instance of the uncertain environment \( \delta \in \Delta \). Practical considerations on this statement are discussed later in the paper.

This problem is *infinite*-dimensional because the design variables are continuous in time, and the feasibility must be imposed on both the time range and the uncertain set. The proposed methodology combines differential flatness and scenario approach to achieve a discretization of the time and the uncertain domain, respectively.

Assume that a set of \( m \) flat outputs, \( \mathbf{q} = Q(x, u, \dot{u}, \ddot{u}, \ldots) \), is available such that

\[
x = \mathbf{X}(q, \dot{q}, \ddot{q}, \ldots),
\]

\[
u = \mathbf{U}(q, \dot{q}, \ddot{q}, \ldots, \delta).
\]

We note that the only mapping \( \mathbf{U} \) is allowed to be non-deterministic, i.e., dependent on \( \delta \).
6 is then recast into

\[ q^* = \arg \left[ \min_{q(t), t \in [0,T]} \left( \max_{\delta \in \Delta} \bar{J}(q, \delta) \right) \right] \quad s.t. : \]
\[ \bar{g}(q, \delta) \leq 0 \quad \forall t \in [0,T] \land \delta \in \Delta \]
\[ \mathbf{X}(q(0), \dot{q}(0), \ddot{q}(0), \ldots) = \mathbf{x}_0 \]
\[ \mathbf{X}(q(T), \dot{q}(T), \ddot{q}(T), \ldots) = 0 \]

where \( \bar{J}(q, \delta) = J(\mathbf{X}(q, \dot{q}, \ddot{q}, \ldots), \delta) \) and \( \bar{g}(q, \delta) = g(\mathbf{X}(q, \dot{q}, \ddot{q}, \ldots), \mathbf{U}(q, \dot{q}, \ddot{q}, \ldots), \delta) \).

This formulation is still infinite-dimensional, but, thanks to differential flatness, the dynamics equations \( \dot{x} = f(x, u, \delta) \) are automatically satisfied and they do not need to be enforced as equality constraints any more, e.g., by means of a pseudospectral transcription.

The discretization in time is performed by expressing the flat outputs in function of a basis of \( l \) sufficiently regular shape functions, \( \Phi(t) = [\phi_1(t), \ldots, \phi_l(t)]^T \):

\[ q^{(i)} = \sum_{j=1}^{l} \phi_j^{(i)}(0)q_j = Q\Phi^{(i)}(t) \tag{9} \]

where the superscripts \((i)\) are the order of the time derivative, and \( Q \in \mathbb{R}^{m \times l} \) is a matrix of coefficients with columns \( q_j \). The minimum regularity of the shape functions is determined by the maximum order of the derivative of \( q(t) \) in the mapping 7.

To remove boundary conditions in Problem 8, the basis \( \Phi \) is projected into its subspace which satisfies them. For this purpose, \( \Phi \) is partitioned into \( l-2n \) independent and \( 2n \) dependent elements* \( \Phi = \begin{bmatrix} \Phi_{ind} \\ \Phi_{dep} \end{bmatrix} \). An analogous partition is performed for the corresponding columns of \( Q \), which is rearranged as \( Q = [Q_{ind}, Q_{dep}] \). The discretized flat output becomes

\[ q^{(i)} = Q_{ind} \tilde{\Phi}^{(i)}(t) + \Psi^{(i)}(t) \tag{10} \]

where

\[ \tilde{\Phi}^{(i)}(t) = \Phi_{ind}^{(i)}(t) - \begin{bmatrix} \Phi_{ind}(0), \dot{\Phi}_{ind}(0), \ldots, \Phi_{ind}(T), \ddot{\Phi}_{ind}(T), \ldots \end{bmatrix} \end{bmatrix}^{-1} \Phi_{dep}^{(i)}(t), \]

\[ \Psi^{(i)}(t) = [q(0), \dot{q}(0), \ldots, q(T), \dot{q}(T), \ldots] \end{bmatrix} \end{bmatrix}^{-1} \Phi_{dep}^{(i)}(t), \]

and \( q(0), \dot{q}(0), \ldots \) and \( q(T), \dot{q}(T), \ldots \) are the boundary conditions of \( q \), and they are solution of \( \mathbf{X}(q(0), \dot{q}(0), \ldots) = \mathbf{x}_0 \) and \( \mathbf{X}(q(T), \dot{q}(T), \ldots) = 0 \), respectively. This is possible because the relationship between states and flat outputs is assumed to be deterministic, and so are the boundary conditions of \( q(t) \).

The time discretization is finalized by limiting the satisfaction of the constraints \( \bar{g} \) to a discrete number \( p \) of check points, \( t_1, \ldots, t_p \). The number and location of the check points can be either

*the choice of dependent functions is arbitrary provided that \( \begin{bmatrix} \Phi_{dep}(0), \Phi_{dep}(0), \ldots, \Phi_{dep}(T), \ddot{\Phi}_{dep}(T), \ldots \end{bmatrix} \) is not singular.
deduced from the properties of the basis, or it can be assessed by picking a number of uniformly-distributed random check points in \([0, T]\) according to Equation 4. In this latter case, the risk \(\epsilon\) should be extremely small. The first option is pursued in this paper.

At this point, Problem 8 is reduced to the semi-infinite approximation

\[
Q_{\text{ind}}^* = \arg \left[ \min_{Q_{\text{ind}} \in \mathbb{R}^{m \times (l-2n)}} \left( \max_{\delta \in \Delta} \tilde{J}(Q_{\text{ind}} \tilde{\Phi}(t) + \Psi(t), \delta) \right) \right] \quad \text{s.t.} \quad \tilde{g} \left( Q_{\text{ind}} \tilde{\Phi}(t_k) + \Psi(t_k), \delta \right) \leq 0 \quad k = 1, \ldots, p; \forall \delta \in \Delta
\]  

(11)

The scenario approach can now be exploited to solve Problem 11. For this purpose, we note that the objective function can be written under the form of Problem 3 by introducing a slack design variable, \(h\), such that

\[
\begin{array}{l}
Q_{\text{ind}}^*, h^* = \arg \left[ \min_{Q_{\text{ind}} \in \mathbb{R}^{m \times (l-2n)}, h \in \mathbb{R}} \left( \max_{\delta \in \Delta} \tilde{J}(Q_{\text{ind}} \tilde{\Phi}(t) + \Psi(t), \delta) \right) \right] \quad \text{s.t.} \quad \tilde{g} \left( Q_{\text{ind}} \tilde{\Phi}(t_k) + \Psi(t_k), \delta \right) \leq 0 \quad k = 1, \ldots, p; \forall \delta \in \Delta \\
\tilde{g} \left( Q_{\text{ind}} \tilde{\Phi}(t_k) + \Psi(t_k), \delta_1 \right) \leq 0 \quad k = 1, \ldots, p; \forall \delta \in \Delta
\end{array}
\]  

(12)

So, given an acceptable risk \(\epsilon\), and a confidence parameter \(\beta\) (small enough to be considered zero for practical purposes), and \(s\) independent samples \((\delta_1, \ldots, \delta_s) \in \Delta\) such that Equation 4 is satisfied (here \(d = (l - 2n)m + 1\)), Problem 11 is recast in the discrete form

\[
Q_{\text{ind}}^* = \arg \left[ \min_{Q_{\text{ind}} \in \mathbb{R}^{m \times (l-2n)}} \left( \max_{\delta \in \{\delta_1, \ldots, \delta_s\}} \tilde{J}(Q_{\text{ind}} \tilde{\Phi}(t) + \Psi(t), \delta) \right) \right] \quad \text{s.t.} \quad \tilde{g} \left( Q_{\text{ind}} \tilde{\Phi}(t_k) + \Psi(t_k), \delta_1 \right) \leq 0 \quad k = 1, \ldots, p
\]  

(13)

With high confidence \(1 - \beta\), the solution of 13, is feasible in a subset \(\Delta_\epsilon \subset \Delta\) such that the probability of \(\Delta_\epsilon\) is no smaller than \(1 - \epsilon\).

**CASE STUDY**

Two case studies are proposed. The first is a rendez-vous maneuver between a fully-actuated chaser and a passive target in an elliptic orbit. The linear periodic equations of Tschauner-Hempel\(^{13}\) are considered as the deterministic plant. The second is aimed at solving the under-actuated problem of a differential-drag-based rendez-vous in a nearly circular orbit. Differential drag is controlled by changing the cross section of the chaser. In-plane Schweighart-Sedwick\(^{14}\) equations are considered for the deterministic system. The relative frame is centered in the target and it is provided with axes \(\hat{x}\) and \(\hat{z}\) pointing toward the outward radial direction and the orbital angular momentum, respectively, and \(\tilde{g}\) completes the right-hand frame. Table 1 lists the simulation parameters for the two case studies.

Providing an accurate and comprehensive identification of the uncertainty sources for these problems is behind the scope of the paper. This is why a reduced set of uncertainty sources is considered to illustrate the methodology. In both cases, uncertainty in the dynamics equation is due to
Table 1. Simulation parameters for the case studies

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Elliptic rendez-vous</th>
<th>Differential drag</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean altitude of perigee</td>
<td>400</td>
<td>350</td>
<td>km</td>
</tr>
<tr>
<td>Mean altitude of apogee</td>
<td>800</td>
<td>350</td>
<td>km</td>
</tr>
<tr>
<td>Mean orbital inclination</td>
<td>30</td>
<td>98</td>
<td>deg</td>
</tr>
<tr>
<td>Mean argument of perigee</td>
<td>0</td>
<td>0</td>
<td>deg</td>
</tr>
<tr>
<td>Mean RAAN</td>
<td>45</td>
<td>45</td>
<td>deg</td>
</tr>
<tr>
<td>Initial mean true anomaly</td>
<td>0</td>
<td>0</td>
<td>deg</td>
</tr>
<tr>
<td>Epoch</td>
<td>1st April 2015</td>
<td>1st April 2015</td>
<td></td>
</tr>
<tr>
<td>Initial relative position</td>
<td>50ξ + 10^3η + 100ηq</td>
<td>20ξ + 10^3ηy + 0ηq</td>
<td>m</td>
</tr>
<tr>
<td>Initial relative velocity</td>
<td>0ξ + 0ηy + 0ηq</td>
<td>0ξ + 0ηy + 0ηq</td>
<td>m/s</td>
</tr>
<tr>
<td>Maneuvering time</td>
<td>12</td>
<td>36</td>
<td>h</td>
</tr>
<tr>
<td>Chaser’s mass</td>
<td>4</td>
<td>4</td>
<td>kg</td>
</tr>
<tr>
<td>Target’s mass</td>
<td>1</td>
<td>2</td>
<td>kg</td>
</tr>
<tr>
<td>Chaser’s cross section</td>
<td>0.01</td>
<td>[0.01-0.03]</td>
<td>m^2</td>
</tr>
<tr>
<td>Target’s cross section</td>
<td>0.01</td>
<td>0.01</td>
<td>m^2</td>
</tr>
<tr>
<td>Maximum thrust</td>
<td>0.05</td>
<td>-</td>
<td>mN</td>
</tr>
<tr>
<td>Nominal drag coefficient</td>
<td>2.8</td>
<td>2.8</td>
<td>-</td>
</tr>
<tr>
<td>Nominal daily solar flux $F_{10.7}$</td>
<td>200</td>
<td>200</td>
<td>sfu</td>
</tr>
<tr>
<td>81-day averaged solar flux $F_{10.7}$</td>
<td>155</td>
<td>155</td>
<td>sfu</td>
</tr>
</tbody>
</table>

the atmospheric force, which is the dominating non-conservative perturbation in low-Earth orbits. Specifically, the uncertainty sources considered are:

- a normal distribution for a correction of the drag coefficient, $\eta$, with nominal value equal to 1 and standard deviation 0.05,

- a normal distribution of the daily solar flux $F_{10.7}$ with standard deviation equal to 10 sfu,

- a constant probability for the outcomes of the geomagnetic indicator $K_p$, which are picked from the event space $[4, 4.3, 4.7, 5, \ldots , 8]$,

- short term density variations modeled with a second-order stationary stochastic process with power spectral density analogous to the one proposed by Zijlstra.\textsuperscript{15}$\textsuperscript{15}$

In the following, this set of uncertainties will be summarized as $\delta$ for the sake of conciseness.

In addition, the quasi-periodic variations of the density experienced throughout the orbit, which are mainly due to changes in the altitude from the ellipsoid and to the daily atmospheric bulge, are modeled by evaluating the NRLMSISE-00 atmospheric model along the trajectory of the target.

Different possibilities exist for the choice of the basis of the flat outputs, $\Phi(t)$. For example, Louembet\textsuperscript{9} uses positive b-splines. This is particularly appealing because of the high flexibility of the b-spline and the possibility to have a reduced support basis in order to achieve a sparse formulation of the optimization problem. In this paper, because of their simple implementation,
truncated Fourier series with fundamental frequency \( f = \frac{1}{2T} \) are used, i.e.,

\[
\Phi^{(i)}(t) = \begin{bmatrix}
0^i \\
(2\pi f)^i \cos (2\pi ft + 0.5\pi i) \\
(2\pi f)^i \sin (2\pi ft + 0.5\pi i) \\
(4\pi f)^i \cos (4\pi ft + 0.5\pi i) \\
(4\pi f)^i \sin (4\pi ft + 0.5\pi i) \\
\vdots
\end{bmatrix}.
\] (14)

The first \( n \) harmonics are retained as the dependent component. The series is truncated when harmonics become smaller than half of the orbital period. A uniform temporal grid of check points with frequency resolution 10 times larger than the highest frequency of the expansion is considered. This is estimated large enough to prevent relevant violation between check points.

The risk and confidence parameters are set to \( \epsilon = 0.1 \) and \( \beta = 10^{-6} \), respectively.

**Rendez-vous in elliptic orbit**

Elliptic rendez-vous is generally tackled by using the linear periodic Tschauner-Hempel equations. These equations are not expressed in the time domain but in the true anomaly of the target, \( \theta \). So, given the vector of the LVLH relative positions and velocities, \( \mathbf{x}(t) \), and the input forces per unit mass, \( \mathbf{u} \), the following change of variables is considered:

\[
\begin{align*}
\ddot{\mathbf{x}}(\theta) &= \begin{bmatrix}
(1 + e \cos \theta) \mathbb{I}_{3 \times 3} & 0 \\
-e \sin \theta \mathbb{I}_{3 \times 3} & (1 + e \cos \theta) \mathbb{I}_{3 \times 3}
\end{bmatrix} \mathbf{x}(t) \\
\ddot{\mathbf{u}} &= \left( \frac{a(1 - e^2)}{\mu} \right)^3 \mathbf{u}
\end{align*}
\] (15)

where \( \mathbb{I}_{3 \times 3} \), \( e \), \( a \), and \( \mu \) are the \( 3 \times 3 \) identity matrix, the eccentricity of the orbit, the semi-major axis and the Earth’s gravitational constant, respectively. The equations of Tschauner-Hempel are

\[
\begin{align*}
\ddot{x}''(\theta) &= 2\dot{y} + \frac{3}{1 + e \cos \theta} \ddot{x} + \ddot{u}_x + \Delta \ddot{F}_{a,x}(\delta, \theta) \\
\ddot{y}''(\theta) &= -2\dot{z} + \ddot{u}_y + \Delta \ddot{F}_{a,y}(\delta, \theta) \\
\ddot{z}''(\theta) &= -\ddot{z} + \ddot{u}_z + \Delta \ddot{F}_{a,z}(\delta, \theta)
\end{align*}
\] (16)

where the apexes indicate the derivative with respect to \( \theta \) and \( \Delta \ddot{F}_{a}(\delta, \theta) \) is the perturbation due to the difference between the aerodynamic force of the chaser and the target. It is obtained from its temporal counterpart, \( \Delta \ddot{F}_{a}(\delta, t) \), with the same transformation of \( \ddot{u} \). The aerodynamic force is computed herein as

\[
\mathbf{F}_{a} = -\frac{1}{2} \rho(\delta, t) \frac{C_d(\delta)S}{m} \|v_{TAS}\| v_{TAS}
\] (17)

where \( \rho \), \( C_d \), \( S \), \( m \), and \( v_{TAS} \) are the atmospheric density, the drag coefficient, the cross section, the mass, and the airspeed, which is equal the vectorial difference between the orbital velocity and the velocity of the co-rotating atmosphere, respectively.

For this system the trivial flat outputs

\[
\mathbf{q} = \begin{bmatrix}
\ddot{x} \\
\ddot{y} \\
\ddot{z}
\end{bmatrix}
\] (18)
are exploited, and the following mapping of the states and controls holds
\[
\begin{align*}
\dot{x} &= \begin{bmatrix} q \\ q' \end{bmatrix} \\
\dot{u} &= q'' + \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} q' + \begin{bmatrix} -\frac{3}{1+e\cos\theta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} q - \Delta \tilde{F}_a(\delta, \theta)
\end{align*}
\]
(19)

The selected objective function for the optimal maneuver consists of:
\[
\tilde{J}(q, \delta) = \sum_{k=1}^{p} \frac{\tilde{u}_{det}(q(\theta_k))^2 + \tilde{u}_{det}(q(\theta_{k+1}))^2}{2} (t_{k+1} - t_k)
\]
(20)

where \( t_k \) and \( \theta_k \) are the time and the corresponding true anomaly of the check points. This objective function is aimed at minimizing the integral of the square of the deterministic component of the control actions, i.e., \( (a (1 - e^2))^{-3} \mu \tilde{u}_{det}(q) \).

The constraints impose that the control action does not exceed the maximum force of the thrusters, \( T_{max} \), so that
\[
-\frac{(a (1 - e^2))^3 T_{max}}{\mu m_c} + \Delta \tilde{F}_a(\delta, \theta_k) \leq \tilde{u}_{det}(q(\theta_k)) \leq \frac{(a (1 - e^2))^3 T_{max}}{\mu m_c} + \Delta \tilde{F}_a(\delta, \theta_k)
\]
(21)

where \( m_c \) is the mass of the chaser.

Figure 2 shows the obtained trajectory and the deterministic component of the control force. Because \( \Delta \tilde{F}_a \) is positive both at the left and right of Equation 21, a larger (positive) value of the differential aerodynamic force results into the reduction of the lower margin for the deterministic component of the control, and the increase of its upper bound. This is clearly emphasized by the crossing of the perigee in the tangential component of the control.

**Rendez-vous using differential drag**

The in-plane Schwaighart-Sedwick equations\(^{14}\) are exploited for this case study
\[
\begin{align*}
\dot{x}(t) &= 2\omega c \dot{y} + (5c^2 - 2) \omega^2 x \\
\dot{y}(t) &= -2\omega c \dot{x} + \Delta F_d(u, \delta, t)
\end{align*}
\]
(22)

where \( \omega \) and \( c = \sqrt{1 + \frac{3J_2}{8a^2} (1 + 3 \cos 2i_{orb})} \), \( i_{orb} \), and \( R_e \) are the constant angular orbital velocity, the Schwaighart-Sedwick coefficient, the orbital inclination, and the Earth’s equatorial radius, respectively. These equations assume a circular reference orbit, secular-only perturbations of the Earth oblateness, \( J_2 \), and differential drag, \( \Delta F_d \), aligned with \( \dot{y} \). In this case, the control variable, \( u \), is the cross section of the chaser, while the cross section of the target is assumed to be constant in time.

The flat output for this system is
\[
q = \frac{x - 2\omega c y}{5c^2 - 2}
\]
(23)
(a) In-plane trajectory. The red arrows are the magnitude and the direction of the deterministic component of the in-plane control forces. The target is at the origin of the axes.

(b) Deterministic component of the control forces per unit mass. The black bounds are the envelop of the worst case perturbations generated by the samples.

Figure 2. Elliptic rendez-vous maneuver.
and the corresponding mapping of the states and controls is

\[
\begin{bmatrix}
\dot{q} \\
\ddot{q} - (5c^2 - 2) \omega^2 q
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{2\omega}{m_c} \\
\frac{q^2 - (5c^2 - 2) \omega^2 \dot{q}}{2\omega c}
\end{bmatrix}
\]

(24)

\[
u = -\frac{2m_c}{\rho(\delta, t) C_{d,c} v^2 T_{AS}} \left( F_{d,t}(\delta, t) + \frac{q^{(4)} - (c^2 - 2) \omega^2 \dot{q}}{2\omega} \right)
\]

where \( F_{d,t} \) and \( C_{d,c} \), are the drag per unit mass of the target and the drag coefficient of the chaser, respectively.

We note that in this case, the only uncertainties in the tangential dynamics equation can be handled. A deterministic mapping between flat output and states cannot be determined otherwise. If perturbations of the radial-direction equation are introduced, the satisfaction of the rendez-vous conditions cannot be deterministically enforced. This is a limitation of the proposed methodology. The physical origin of this restriction is that the system is under-actuated. So, given two independent disturbances in the radial and in-track equations of motion, respectively, the single control action cannot simultaneously compensate them in order to give the desired deterministic acceleration on the two axes.

The selected objective function for the optimal maneuver consists of the integral of the differential drag:

\[
\tilde{J}(q, \delta) = \frac{1}{p-1} \sum_{k=1}^{p-1} \left( q^{(4)}(t_k) - (c^2 - 2) \omega^2 \dot{q}(t_k) \right)^2 + \left( q^{(4)}(t_{k+1}) - (c^2 - 2) \omega^2 \dot{q}(t_{k+1}) \right)^2
\]

\[
\frac{1}{2(2\omega c)^2} (t_{k+1} - t_k)
\]

(25)

where \( t_k \) are the evaluating time of the check points. This objective function is aimed at maximizing the margin of maneuverability during the on-line tracking of the reference path.

The constraints impose that the differential drag is bounded by the minimum and maximum cross section of the chaser, \( S_{c,min} \) and \( S_{c,max} \), respectively. It follows

\[
\Delta F_d(S_{c,\text{max}}, \delta, t_k) \leq \frac{z^{(4)}(t_k) - (c^2 - 2) \omega^2 \dot{z}(t_k)}{2\omega c} \leq \Delta F_d(S_{c,\text{min}}, \delta, t_k)
\]

\[
k = 1, \ldots, p.
\]

(26)

Figure 3 shows the obtained trajectory and the deterministic component of the control force. In this case, the envelop of the worst-case samples is due to the values of \( \delta \) which minimize the absolute value of the drag. The smaller the drag the smaller the control authority of the differential drag.

**THEORETICAL RESULTS AND PRACTICAL CONSIDERATIONS**

The main outcome of the solution of Problem 13 is a reference trajectory with guaranteed robustness in \( \Delta \epsilon \). This means that with probability no smaller than \( 1 - \epsilon \), the trajectory can be theoretically tracked, i.e., without saturating the control actuators or the differential drag. We note that the \( \epsilon \)-risk warranty is valid for the semi-infinite Problem 11. This means that possible violation of the constraints might occur between check points. However, if the location of the check points is
(a) In-plane trajectory. The color indicates the magnitude of the differential drag. Blue is minimum, red maximum. The target is at the origin of the axes.

(b) Differential drag per mass unit. The black bounds are the envelop of the worst case perturbations generated by the samples.

Figure 3. Rendez-vous maneuver using differential drag.
Figure 4. Monte Carlo estimation of the feasible set of the optimal solution for the differential drag rendez-vous. Bias and solar activity cut. Blue dots and red crosses are the feasible and unfeasible samples, respectively.

sufficiently dense, it is not problematic. Possibly, an investigation of the obtained solution should be carried out to assess that significant violation between consecutive check points does not occur.

Though the solution is $\epsilon$-risk warranted, the scenario approach does not provide with direct insight in the topology of $\Delta_\epsilon$. This can be revealed by a posteriori evaluation of the obtained solution. Figure 4 shows the $F_{10.7}$-$\eta$ cut of the feasibility analysis of the Monte Carlo evaluation of the solution for the differential drag case study. The only $2.5\%(<10\% \approx \epsilon)$ of the samples is not feasible. The unfeasible set corresponds to the values of the parameters which minimize the drag force, i.e., bottom-left corner. This result is expected because the smaller the drag, the smaller the authority of the differential drag. On the contrary, $\Delta_\epsilon$ for the elliptic rendez-vous maneuver is located in the center of the diagram. In fact, according to Equation 21, the upper and lower bounds of the deterministic component of the control (black curves of Figure 2(b)) force are due to the envelop of the minimum and maximum values of the differential drag, respectively.

Concerning the real-time implementation of the maneuver, the theoretical result holds only if direct knowledge of the disturbance is available. However, most often, no information is available on the disturbances. For this purpose, feedback compensation of the tracking error of the reference trajectory is mandatory. The compensator should be designed in order to efficiently filter the frequency content of $\delta$. An interesting approach for robust tracking based on differential algebra is proposed by Di Lizia. Another alternative is to use, once again, the scenario approach. Campi et al. provide some examples of the design of robust feedback controllers using the scenario approach. Uncertainty in the initial conditions must also be handled by the on-line compensator.

CONCLUSION

We proposed an optimal control approach for robust rendez-vous planning which combines differential flatness and the scenario approach. The exploitation of differential flatness allows the removal of dynamics equations and boundary conditions from the problem. This makes the scenario approach applicable. The scenario approach is then exploited to achieve a solution with guaranteed robustness.

One significant advantage of this approach is that the planned trajectory can be theoretically
tracked for a prescribed portion of the possible outcomes of the uncertainties.

The proposed methodology is very flexible in terms of the complexity of the stochastic environment, which can include random variables, stochastic processes, and random fields. A probabilistic characterization of the uncertain set is not required.

A possible limitation of the methodology is the existence and the determination of the flat outputs. Furthermore, uncertainty sources that affect the deterministic mapping between flat outputs and states cannot be handled.

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